

PLETHYSTIC SUBSTITUTION

GEORGE H. SEELINGER

1. INTRODUCTION

In [Mac79, p 135], a new type of product on symmetric functions is introduced called “plethysm,” which allows one to take $f \in \Lambda^m$ and $g \in \Lambda^n$ to get a product $f[g] \in \Lambda^{mn}$ (denoted $f \circ g$ in [Mac79]). This notion has become increasingly prevalent in algebraic combinatorics research, and this monograph seeks to give an outline of some of the essentials.

2. DEFINITION AND PROPERTIES

Departing from [Mac79], we define the following.

2.1. Definition. Given a Laurent series A in indeterminates a_1, a_2, a_3, \dots , we define $p_n[A]$ to be the series where each a_i is changed to a_i^n . In other words, each indeterminate is raised to the n th power. In particular, given a symmetric function $g \in \Lambda$, $p_n[g(x_1, x_2, \dots)] = g(x_1^n, x_2^n, \dots)$.

Furthermore, it is a common convention to let $X = x_1 + x_2 + x_3 + \dots$ and then write things such as

$$p_n[X] = p_n(x_1, x_2, x_3, \dots)$$

2.2. Example. (a) If $A = a_1 + a_2 + a_3 + \dots$, then $p_n[A] = a_1^n + a_2^n + a_3^n + \dots$.
(b) In particular, $p_n[p_m] = (x_1^n)^m + (x_2^n)^m + \dots = p_{nm} = p_m[p_n]$. Thus, $p_n[1] = 1$.

2.3. Proposition. [Mac79, p 135] *For $n \geq 1$, the mapping $g \mapsto p_n[g]$ is an endomorphism of the ring Λ .*

Next, since any $f \in \Lambda$ can be written as a (rational) linear combination of p_λ 's and each p_λ is a product of p_n 's, we extend the definition of plethysm to say

2.4. Definition. Given a Laurent series A ,

- (a) we say $p_\lambda[A] = p_{\lambda_1}[A]p_{\lambda_2}[A] \cdots p_{\lambda_\ell}[A]$ and
- (b) $(f + g)[A] = f[A] + g[A]$ for any $f, g \in \Lambda$, and

Thus, we can compute $f[A]$ for any symmetric function $f \in \Lambda$ by writing it as a linear combination of p_λ 's and evaluating the plethysm on each term.

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2.5. **Example.** (a) Given $A = 1 + t + t^2 + t^3 + \dots = \frac{1}{1-t}$, we get $f[\frac{1}{1-t}] = f(1, t, t^2, t^3, \dots)$ since

$$p_n \left[\frac{1}{1-t} \right] = 1 + t^n + t^{2n} + \dots = p_n(1, t, t^2, \dots)$$

- (b) Recall $p_1(x) = x_1 + x_2 + \dots =: X$. Then, $f[X + a]$ adds a variable a to our set of variables. Similarly, $f[X - x_i]$ removes x_i from the set of variables.
- (c) Combining the ideas above, $f[X - (1-t)x_i]$ removes variable x_i but replaces it with variable tx_i .
- (d) Finally, $f[\frac{1}{1-t}] = f(1, t, t^2, \dots)$ and $f[\frac{X}{1-t}] = f(x_1, tx_1, t^2x_1, \dots, x_2, tx_2, t^2x_2, \dots)$ since $\frac{X}{1-t} = x_1 + tx_1 + t^2x_1 + \dots + x_2 + tx_2 + t^2x_2 + \dots$.

2.6. **Proposition.** Given $c \in \mathbb{Q}$, we get, by definition, that $f[cA] = cf[A]$ for all $f \in \Lambda$ and Laurent series A . However, given an indeterminate t , we get $p_n[tA] = t^n p_n[A]$. In other words, plethysm and variable evaluation do not commute.

Proof. This follows since plethysm affects indeterminates but not constants. □

2.7. **Remark.** The proposition above can be the source of much confusion. One way to distinguish between these two different kinds of values is to call the constants *binomial variables*. So, in the proposition above, we say that c is a binomial variable but t is not.

2.8. **Definition.** It can be convenient to introduce a minus sign to each variable in the plethystic substitution. So, we define the variable ϵ such that

$$p_r[\epsilon X] := p_r(-x_1, -x_2, -x_3, \dots) = (-1)^r p_r[X]$$

where $X = x_1 + x_2 + x_3 + \dots$.

2.9. **Remark.** Notice that $p_r[\epsilon X]$ is not necessarily equal to $p_r[-X]$ in our notation. In particular, for a binomial variable $c \in \mathbb{Q}$, we have

$$p_r[cX] = cp_r[X] \text{ but } p_r[\epsilon X] = (-1)^r p_r[X]$$

Furthermore, authors are often not careful with this distinction, so one needs to use context.

2.10. **Proposition.** [Mac79, p 135] *Plethysm is associative. That is,*

$$(f[g])[h] = f[g[h]]$$

Proof. Because the p_n generate Λ over \mathbb{Q} , we need only verify the associativity for p_n 's, which we already did in 2.2. □

2.11. **Lemma.** Given Laurent series A and B , we get

$$p_k[A + B] = p_k[A] + p_k[B]$$

Proof. By definition, $p_k[A + B]$ raises all the indeterminates from A and B to the k th power, which is the same effect as $p_k[A]$ and $p_k[B]$. \square

Now, recall the Cauchy kernel

$$\Omega(x, y) = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

We seek to generalize this notion as follows. Let us define

$$\Omega := \exp \left(\sum_{k=1}^{\infty} \frac{p_k}{k} \right)$$

which gives us that

2.12. Proposition. (a)

$$\Omega[x] = \exp \left(\sum_{k=1}^{\infty} \frac{x^k}{k} \right) = \exp(\log(x-1)) = \exp(\log((1-x)^{-1})) = \frac{1}{1-x}$$

(b) $\Omega[A + B] = \Omega[A]\Omega[B]$ and $\Omega[-A] = \frac{1}{\Omega[A]}$ for any Laurent series A and B

(c)

$$\Omega[X] = \prod_{i \geq 1} \frac{1}{1 - x_i} \text{ and } \Omega[XY] = \Omega(x, y)$$

for formal power series $X = \sum x_i$ and $Y = \sum y_j$.

Proof. By definition, $p_k[x] = x^k$ and so the first part follows. For part (b), using the lemma above, we have

$$\exp(p_k[A + B]) = \exp(p_k[A] + p_k[B]) = \exp(p_k[A]) \exp(p_k[B])$$

and so $\Omega[A + B] = \Omega[A]\Omega[B]$. Similarly,

$$\exp(p_k[-A]) = \exp(-p_k[A]) = \frac{1}{\exp(p_k[A])}$$

Finally, part (c) follows from repeated iteration of part (a). \square

2.13. Corollary. (a) $e_r[X] = h_r[-\epsilon X] = (-1)^r h_r[-X]$.

(b) The involution on symmetric functions $\omega: \Lambda \rightarrow \Lambda$ corresponds to the plethystic substitution $X \mapsto -\epsilon X$.

Proof. To start, we note $\Omega[tX] = \prod_{i \geq 1} \frac{1}{1 - tx_i} = \sum_{r \geq 0} h_r[X] t^r$ and

$$\sum_{r \geq 0} h_r[-\epsilon X] t^r = \Omega[-t\epsilon X] = \prod_{i \geq 1} 1 + x_i = \sum_{r \geq 0} e_r[X] t^r.$$

Then, the first part follows immediately. The second part follows from the first since one definition of ω is precisely that $\omega(h_r) = e_r$. \square

3. EXAMPLES WITH SCHUR FUNCTIONS

Some examples

3.1. Proposition. [Sta99, Cor 7.21.3] *We have*

$$s_\lambda \left[\frac{1}{1-t} \right] = s_\lambda(1, t, t^2, t^3, \dots) = \frac{t^{n(\lambda)}}{\prod_{x \in \lambda} (1 - t^{h(x)})}$$

for $h(x)$ the hook-length of cell $x \in \lambda$ and $n(\lambda) = \sum_i (i-1)\lambda_i$.

Proof. As discussed above, $f \left[\frac{1}{1-t} \right] = f(1, t, t^2, t^3, \dots)$ for any symmetric function f . Now, we observe that

$$s_\lambda(1, t, t^2, t^3, \dots, t^{n-1}) = \frac{t^{n(\lambda)+n(n-1)(n-2)/6} \prod_{i < j} (1 - t^{\lambda_i - \lambda_j - i + j})}{t^{n(n-1)(n-2)/6} \prod_{i < j} (1 - t^{-i+j})}$$

However, one can show using the combinatorics of tableaux (see [Man98, Exercise 1.4.9] and proof of [Man98, Proposition 1.4.10]) that

$$\prod_{x \in \lambda} (1 - t^{h(x)}) \prod_{i < j} (1 - t^{\lambda_i - \lambda_j - i + j}) = \prod_{i=1}^n \prod_{k=1}^{\lambda_i + n - i} (1 - t^k)$$

and so, plugging this in, we get

$$s_\lambda(1, t, t^2, \dots, t^{n-1}) = t^{n(\lambda)} \frac{\prod_{i=1}^n \prod_{k=1}^{\lambda_i + n - i} (1 - t^k)}{\prod_{x \in \lambda} (1 - t^{h(x)}) \prod_{i=1}^n \prod_{k=1}^{n-i} (1 - t^k)} = t^{n(\lambda)} \frac{\prod_{i=1}^n \prod_{k=n-i+1}^{\lambda_i + n - i} (1 - t^k)}{\prod_{x \in \lambda} (1 - t^{h(x)})}$$

However, since $\lambda_i = 0$ for $i > \ell(\lambda)$, we can remove one dependence on n to get:

$$s_\lambda(1, t, t^2, \dots, t^{n-1}) = t^{n(\lambda)} \frac{\prod_{i=1}^{\ell} \prod_{k=n-i+1}^{\lambda_i + n - i} (1 - t^k)}{\prod_{x \in \lambda} (1 - t^{h(x)})}$$

At this point, [Sta99] claims that $\lim_{n \rightarrow \infty} (1 - t^n) = 1$, so we are done. □

3.2. Proposition. *Let λ be a partition. Then,*

$$s_\lambda[X + a] = \sum_k a^k \sum_{\lambda = \nu + \text{horizontal } k\text{-strip}} s_\nu(x)$$

for $X = x_1 + x_2 + x_3 + \dots$.

Proof. Using Littlewood's combinatorial description of Schur functions, we get

$$s_\lambda(x_1, \dots, x_n) = \sum_{T \in \text{SSYT}(\lambda)} x^{\text{wt}(T)}$$

However, since semistandard tableaux must have strictly increasing columns, all the boxes labelled n must form a (possibly empty) horizontal strip. Thus,

Of course, I do not understand why this should be true; certainly, it would not work for my calculus students. Perhaps since we are using an expansion where $t \nmid 1$ anyways, this follows.

if we break up the sum based on how many boxes labelled n there are, we get

$$s_\lambda(x_1, \dots, x_n) = \sum_{k \geq 0} x_n^k \sum_{\lambda = \nu + \text{horizontal } k\text{-strip}} \sum_{\Gamma \in \text{SSYT}(\nu)} x^{\text{wt}(\Gamma)}$$

where $\text{SSYT}(\nu)$ are labelled with letters $\{1, \dots, n-1\}$.

$$= \sum_{k \geq 0} x_n^k \sum_{\lambda = \nu + \text{horizontal } k\text{-strip}} s_\nu(x_1, \dots, x_{n-1})$$

Thus, we see how to write a Schur function in terms of Schur functions with one fewer variable. \square

3.3. Proposition. [Mac79, 8.8] *Given a partition λ and symmetric functions g, h ,*

$$s_\lambda[g + h] = \sum_{\mu, \nu} c_{\mu\nu}^\lambda s_\mu[g] s_\nu[h]$$

where $c_{\mu\nu}^\lambda$ are the Littlewood-Richardson coefficients.

Proof. This follows from [Mac79, 5.9] which states

$$s_\lambda(x, y) = \sum_{\mu, \nu} c_{\mu\nu}^\lambda s_\mu(y) s_\nu(x)$$

following from formal manipulations of skew-Schur functions. \square

3.4. Remark. One can actually take this as the definition of a skew-Schur function. That is, we can define the skew-Schur function $s_{\lambda/\mu}$ to be such that

$$s_\lambda[X + Y] = \sum_{\mu} s_{\lambda/\mu}[X] s_\mu[Y]$$

for $X = x_1 + x_2 + \dots$ and $Y = y_1 + y_2 + \dots$.

Finally, we state without proof

3.5. Theorem. *Given partitions λ, μ , we get*

$$s_\lambda[s_\mu] = \sum_{\nu} a_{\lambda\mu}^\nu s_\nu$$

with $a_{\lambda\mu}^\nu \in \mathbb{Z}_{\geq 0}$.

3.6. Remark. While one can prove that these coefficients are non-negative (see [Mac79, Appendix I.A]), actually describing these coefficients is an old and difficult problem in general, sometimes referred to as the ‘‘plethysm problem.’’

REFERENCES

- [Hai03] M. Haiman, *Combinatorics, Symmetric Functions, and Hilbert Schemes* (2003). <https://math.berkeley.edu/~mhaiman/ftp/cdm/cdm.pdf>.
- [Mac79] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 1979. 2nd Edition, 1995.
- [Man98] L. Manivel, *Symmetric Functions, Schubert Polynomials, and Degeneracy Loci*, 1998. Translated by John R. Swallow; 2001.
- [Sch14] A. Schilling, *Plethystic notation*, 2014. UCD MAT 280: Macdonald Polynomials and Crystal Bases; https://math.libretexts.org/LibreTexts/University_of_California%2C_Davis/UCD_MAT_280%3A_Macdonald_Polynomials_and_Crystal_Bases/Plethystic_notation..
- [See18] G. H. Seelinger, *Algebraic Combinatorics*, 2018. [Online] <https://ghseeli.github.io/grad-school-writings/class-notes/algebraic-combinatorics.pdf>.
- [Sta99] R. P. Stanley, *Enumerative Combinatorics, Vol 2*, 1999.