NOTES ON QUANTUM GROUPS

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1. Basics

1.1. Definitions.

1.1. Definition. For an indeterminate q and $n \in \mathbb{Z}$, we define

(a) $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$
(b) $[0]_q! := 1$ and $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$ for $n \in \mathbb{Z}_{\geq 0}$
(c) If $m \geq n \geq 0$, then

$$\binom{m}{n}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}$$

1.2. Proposition. *We have the identity*

$$\binom{m+1}{n}_q = q^n \binom{m}{n}_q + q^{-m+n-1}\binom{m}{n-1}_q$$

and that both $[n]_q$ *and* $\binom{m}{n}_q$ *are elements of* $\mathbb{Z}[q, q^{-1}]$

Proof. Observe that

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{q}{q^n} \cdot \frac{q^{2n} - 1}{q^2 - 1} = \frac{1}{q^{n-1}}(q^{2n-2} + q^{2n-4} + \cdots + q^2 + 1) \in \mathbb{Z}[q, q^{-1}]$$

This immediately gives that $[n]_q! \in \mathbb{Z}[q, q^{-1}]$. In □

The idea of these definitions is that they are a "q-deformation" of the integer n and the binomial expression. To see this idea, we remark that

$$[n]_q \to n \text{ and } \binom{m}{n}_q \to \binom{m}{n} \text{ as } q \to 1$$

1.3. Definition. Throughout, let $(A, \Pi, \Pi^\vee, P, P^\vee)$ be Cartan associated to A where A is a symmetrizable generalized Cartan matrix with a symmetrizing matrix $D = \operatorname{diag}(s_i \in \mathbb{Z}_{\geq 0} \mid i \in I)$.

1.4. Definition. The *quantum group* $\mathcal{U}_q(\mathfrak{g})$ associated with Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the associative algebra with 1 over $F(q)$ generated by elements e_i, f_i for $i \in I$ and K_μ for $\mu \in P^\vee$ with the following relations

(a) $K_0 = 1, K_\mu K_{\mu'} = K_{\mu+\mu'}$ for $\mu, \mu' \in P^\vee$

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(b) $K_\mu e_i K_{-\mu} = q^{\alpha_i(\mu)} e_i$ for $\mu \in P^\vee$
(c) $K_\mu f_i K_{-\mu} = q^{-\alpha_i(\mu)} f_i$ for $\mu \in P^\vee$
(d) $e_i f_j - f_j e_i = \delta_{ij} \frac{K_{s_i h_i} - K_{-s_i h_i}}{q^{s_i} - q^{-s_i}}$ for $i, j \in I$
(e) $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{K_{s_i}} e^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$
(f) $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{K_{s_i}} f^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$

1.5. **Example.** Let $\mathfrak{g} = \mathfrak{sl}_2 = \langle e, f, h \rangle$. Then, we will write $K := K_h$ since $P^\vee$ is spanned only by h. Thus, $\mathcal{U}_q(\mathfrak{sl}_2) = \langle e, f, K, K^{-1} \rangle$. Since the Cartan matrix for $\mathfrak{sl}_2$ is just $A = (2)$, which is already symmetric, we will take $D = Id$. Then, many relations simplify. For instance,

$$KeK^{-1} = qe, KfK^{-1} = q^{-1} f, ef - fe = \frac{K - K^{-1}}{q - q^{-1}}$$

1.6. **Example.** Now, consider $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$, which has Cartan matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

which will have generators

$$\mathcal{U}_q(\hat{\mathfrak{sl}}_2) = \langle e_0, e_1, f_0, f_1, K_0^{\pm}, K_1^{\pm} \rangle$$

where we lazily encode $K_i := K_{h_i}$ and relations

(a)
$$\begin{cases} K_i^{\pm} e_i K_i^{\mp} = q^2 e_i & i = 0, 1 \\ K_i^{\pm} e_j K_i^{\mp} = q^{-2} e_i & i \neq j \end{cases}$$

(b)
$$\begin{cases} K_i^{\pm} f_i K_i^{\mp} = q^{-2} f_i & i = 0, 1 \\ K_i^{\pm} f_j K_i^{\mp} = q^2 f_i & i \neq j \end{cases}$$

(c)
$$\begin{cases} e_i f_j = f_j e_i & i \neq j \\ e_i f_i - f_i e_i = \frac{K_i - K_i^{-1}}{q - q^{-1}} & i = 0, 1 \end{cases}$$

(d)
$$\begin{cases} e_0^3 e_1 - (q^2 + 1 + q^{-2}) e_0^2 e_1 e_0 + (q^2 + 1 + q^{-2}) e_0 e_1 e_0^2 - e_1 e_0^3 = 0 \\ e_1^3 e_0 - (q^2 + 1 + q^{-2}) e_1^2 e_0 e_1 + (q^2 + 1 + q^{-2}) e_1 e_0 e_1^2 - e_0 e_1^3 = 0 \end{cases}$$

$$\implies \begin{cases} e_0^3 e_1 = e_1 e_0^3 + (q^2 + 1 + q^{-2}) e_0^2 e_1 e_0 - (q^2 + 1 + q^{-2}) e_0 e_1 e_0^2 \\ e_1^3 e_0 = e_0 e_1^3 + (q^2 + 1 + q^{-2}) e_1^2 e_0 e_1 - (q^2 + 1 + q^{-2}) e_1 e_0 e_1^2 \end{cases}$$

1.7. **Definition.** In order to simplify notation, we will write

(a) $q_i := q^{s_i}$
(b) $K_i := K_{s_i h_i}$
(c) For $\alpha = \sum_i \eta_i \alpha_i$ a root, we will write $K_\alpha := \prod_i K_i^{\eta_i}$.

1.8. **Proposition.** *If we set* $\deg f_i = -\alpha_i, \deg K_h = 0$, *and* $\deg e_i = \alpha_i$, *then we get the* root space decomposition

$$\mathcal{U}_q(\mathfrak{g}) = \bigoplus_{\alpha \in \Phi} (\mathcal{U}_q)_\alpha$$

where $(\mathcal{U}_q)_\alpha := \{u \in \mathcal{U}_q(\mathfrak{g}) \mid K_h u K_{-h} = q^{\alpha(h)} u$ *for all* $h \in P^\vee\}$.

Proof. The defining relations for the quantum group are all homogeneous with respect to our choice of degree, so we can write $\mathcal{U}_q(\mathfrak{g})$ as a direct sum by degree. □

1.9. **Definition.** Define the *quantum adjoint operators* by

$$(\operatorname{ad}_q x)(y) := xy - q^{(\alpha|\beta)} yx$$

for $x \in (\mathcal{U}_q)_\alpha, y \in (\mathcal{U}_q)_\beta$, and $\alpha, \beta \in \Phi$ where $(\cdot|\cdot) \colon \mathfrak{h} \times \mathfrak{h} \to F$ is the nondegenerate bilinear form defined by

$$\begin{cases} (h_i|h) := \alpha_i(h)/s_i & \text{for } h \in \mathfrak{h} \\ (d_s|d_t) := 0 & \text{for } s, t = 1, \ldots, |I| - \operatorname{rank} A \end{cases}$$

and then extend the definition by linearity to the entire quantum group.

1.10. **Lemma.** *We get*

$$(\operatorname{ad}_q e_i)^N (e_j) = \sum_{k=0}^{N} (-1)^k q_i^{k(N+a_{ij}-1)} \binom{N}{k}_{q_i} e_i^{N-k} e_j e_i^k$$

1.11. **Remark.** The last two defining relation of the quantum group are called the *quantum Serre relations*. Using the lemma above, for $i \neq j$, we can rewrite them as

(a) $(\operatorname{ad}_q e_i)^{1-a_{ij}}(e_j) = 0$
(b) $(\operatorname{ad}_q f_i)^{1-a_{ij}}(f_j) = 0$

1.12. **Proposition.** [HK02, Prop 3.1.2] *The quantum group* $\mathcal{U}_q(\mathfrak{g})$ *has a Hopf algebra structure given by*

(a) $\Delta(K_h) = K_h \otimes K_h$
(b) $\Delta(e_i) = e_i \otimes K_{s_i h_i}^{-1} + 1 \otimes e_i$
(c) $\Delta(f_i) = f_i \otimes 1 + K_{s_i h_i} \otimes f_i$
(d) $\epsilon(K_h) = 1, \epsilon(e_i) = \epsilon(f_i) = 0$
(e) $S(K_h) = K_{-h}, S(e_i) = -e_i K_{s_i h_i}, S(f_i) = -K_{s_i h_i}^{-1} f_i$

for $h \in P^\vee$ *and* $i \in I$.

1.13. **Definition.** Given quantum group $\mathcal{U}_q(\mathfrak{g})$, we define $\mathcal{U}_q^+$ (respectively $\mathcal{U}_q^-$) to be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements e_i (respectively $\mathcal{U}_q^-$) for $i \in I$. We also define $\mathcal{U}_q^0$ to be the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by K_h for $h \in P^\vee$.

1.14. **Theorem.** *We have the* triangular decomposition

$$\mathcal{U}_q(\mathfrak{g}) \cong \mathcal{U}_q^- \otimes \mathcal{U}_q^0 \otimes \mathcal{U}_q^+$$

Proof. We postpone the proof until we can establish the machinery to prove it below. □

1.15. **Definition.** Let $T\colon \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g})$ be a linear map defined by

$$T(K_h) = K_{-h}, T(e_i) = f_i, T(f_i) = e_i$$

for $h \in P^\vee, i \in I$. Furthermore, let

$$\sigma(a \otimes b) := b \otimes a$$

for $a, b \in \mathcal{U}_q(\mathfrak{g})$.

1.16. **Proposition.** *T is an algebra endomorphism on $\mathcal{U}_q(\mathfrak{g})$. Furthermore,*

(a) $T^2 = id$
(b) $\Delta \circ T = \sigma \circ (T \otimes T) \circ \Delta$
(c) $T(\mathcal{U}_q^+) = U_q^-$ *inducing an algebra isomorphism between* $\mathcal{U}_q^+$ *and* $\mathcal{U}_q^-$*.*

Proof. To prove this, we check the results on each of the generators.

(a)

$$T^2(K_h) = T(K_{-h}) = K_h, T^2(e_i) = T(f_i) = e_i, T^2(f_i) = T(e_i) = f_i$$

(b) We check

$$\begin{aligned}
(\Delta \circ T)(K_h) &= K_{-h} \otimes K_{-h} \\
&= \sigma(K_{-h} \otimes K_{-h}) \\
&= \sigma(T(K_h) \otimes T(K_h)) \\
&= (\sigma \circ (T \otimes T) \circ \Delta)(K_h) \\
(\Delta \circ T)(e_i) &= f_i \otimes 1 + K_{s_i h_i} \otimes f_i \\
&= \sigma(1 \otimes f_i + f_i \otimes K_{s_i h_i}) \\
&= \sigma(T(1) \otimes T(e_i) + T(e_i) \otimes T(K_{-s_i h_i})) \\
&= (\sigma \circ (T \otimes T))(e_i \otimes K_{s_i h_i}^{-1} + 1 \otimes e_i) \\
&= \sigma \circ (T \otimes T) \circ \Delta
\end{aligned}$$

And the computation $(\Delta \circ T)(f_i)$ works similarly to the above.

(c) This part is immediate since $T(e_i) = f_i \in \mathcal{U}_q^-$ and $T^2 = id$, so T is, in particular, bijective. Thus, since T is an algebra homomorphism by construction, it gives an algebra isomorphism.

□

1.17. **Lemma.** [HK02, Lemma 3.1.4]

(a) $\mathcal{U}_q^{\geq 0} \cong \mathcal{U}_q^0 \otimes \mathcal{U}_q^+$
(b) $\mathcal{U}_q^{\leq 0} \cong \mathcal{U}_q^- \otimes \mathcal{U}_q^0$

Proof. The proof is quite technical but for item (b) consists of

(i) Let $\{f_\xi\}_{\zeta\in\Omega}$ be a basis of $\mathcal{U}_q^-$ consisting of monomials in f_i's.
(ii) Show the map $\phi\colon \mathcal{U}_q^- \otimes \mathcal{U}_q^0 \to \mathcal{U}_q^{\geq 0}$ given by $\phi(f_\zeta \otimes K_h) = f_\zeta K_h$ is surjective.
(iii) Show the set $\{f_\zeta K_h \mid \zeta \in \Omega, h \in P^\vee\}$ is linearly independent over $F(q)$ by directly using the definition of linear independence and the comultiplication map.

Part (a) proceeds analogously (presumably). □

Proof of 1.14. [HK02, p 42] Let $\{f_\zeta\}_{\zeta\in\Omega}$ and $\{e_\zeta\}_{\zeta\in\Omega}$ be monomial bases of $\mathcal{U}_q^-$ and $\mathcal{U}_q^+$, respectively. It suffices to show $\{f_\zeta K_h e_\eta \mid \zeta, \eta \in \Omega, h \in P^\vee\}$ is linearly independent over $F(q)$.

Let $C_{\zeta,h,\eta} \in F(q)$ be scalar such that

$$\sum_{\zeta,h,\eta} C_{\zeta,h,\eta} f_\zeta K_h e_\eta = 0$$

Using the root space decomposition of $\mathcal{U}_q(\mathfrak{g})$ 1.8, we have

$$\sum_{\substack{h\in P^\vee \\ \deg f_\zeta + \deg e_\eta = \gamma}} C_{\zeta,h,\eta} f_\zeta K_h e_\eta = 0, \forall \gamma \in Q$$

However, if we apply the comultiplication map to this expression, we get

$$0 = \sum_{\substack{h\in P^\vee \\ \deg f_\zeta + \deg e_\eta = \gamma}} C_{\zeta,h,\eta}(f_\zeta \otimes 1 + \cdots + K_{-\deg f_\zeta} \otimes f_\zeta)(K_h \otimes K_h)(e_\eta \otimes K_{\deg e_\eta}^{-1} + \cdots + 1 \otimes e_\eta)$$

Then, using the partial ordering on $P^\vee$ given by $\lambda \leq \mu \iff \mu - \lambda \in Q_+$, we can pick $\alpha = \deg f_\zeta$ minimal and $\beta = \deg e_\eta$ maximal among those for which $C_{\zeta,h,\eta}$ is nonzero. Then,

$$\sum_{\substack{h\in P^\vee \\ \deg f_\zeta = \alpha, \deg e_\eta = \beta}} C_{\alpha,h,\beta} C_{\zeta,h,\eta}(f_\zeta K_h \otimes K_h e_\eta) = 0$$

However, the vectors $\{f_\zeta K_h\}_{\zeta,h}$ are linearly independent by the lemma above, as are $\{e_\eta K_h\}_{\eta,h}$. Thus, we get $C_{\zeta,h,\eta} = 0$. □

1.2. **Representation theory of quantum groups.** The representation theory of quantum groups has similar results to that of the representation theory of Kac-Moody algebras.

This section has no proofs. To be filled in later, maybe when I take a quantum groups course.

1.18. **Definition.** (a) We say a $\mathcal{U}_q(\mathfrak{g})$-module V^q is a *weight module* if it admits a weight space decomposition

$$V^q = \bigoplus_{\mu\in P} V_\mu^q \text{ where } V_\mu^1 = \{v \in V^q \mid K_h v = q^{\mu(h)} v \text{ for all } h \in P^\vee\}$$

(b) We say a vector $v \in V^q$ is a *weight vector* of weight $\mu \in P$ if $K_h v = q^{\mu(h)} v$ for all $h \in P^\vee$.

(c) We say a weight vector $v \in V^q$ of weight μ is a *maximal vector* if $e_i v = 0$ for all $i \in I$.
(d) If $V^q_\mu \neq 0$, μ is called a *weight* of V^q and V^q_μ is called the *weight space* of weight $\mu \in P$. The set of all weights of V^q will be denoted $\mathrm{wt}(V^q)$.
(e) The *weight multiplicity* of μ in V^q is $\dim V^q_\mu$.
(f) If $\dim V^q_\mu < \infty$ for all $\mu \in \mathrm{wt}(V^q)$, then the *character* of V^q is given by

$$\operatorname{ch} V^q = \sum_\mu \dim V^q_\mu e^\mu$$

where e^μ are formal basis elements of the group algebra $F[P]$ with multiplication $e^\lambda e^\mu = e^{\lambda+\mu}$.

1.19. **Proposition.** [HK02, Proposition 3.2.1] *Every submodule of a weight module over $\mathcal{U}_q(\mathfrak{g})$ is also a weight module.*

1.20. **Definition.** (a) For $\lambda \in P$, we set

$$D(\lambda) := \{\mu \in P \mid \mu \leq \lambda\}$$

(b) The *category* $\varnothing^q$ consists of all weight modules V^q over $\mathcal{U}_q(\mathfrak{g})$ with finite dimensional weight spaces for which there exist a finite number of elements $\lambda_1, \ldots, \lambda_s \in P$ such that

$$\mathrm{wt}(V^q) = D(\lambda_1) \cup \cdots \cup D(\lambda_s)$$

(c) A weight module V^q is called a *highest weight module* with *highest weight* λ if there exists weight vector of weight λ, say $0 \neq v_\lambda \in V^q$, such that v_λ is a maximal vector and $V^1 = \mathcal{U}_q(\mathfrak{g})v_\lambda$. In this setting, the vector v_λ is unique up to constant multiple and is called the *highest weight vector*.

1.21. **Proposition.** [HK02]*pp 43–44 Given V^q a highest weight module with highest weight λ,*

(a) $V^q = \mathcal{U}_q^- v_\lambda$ *by the triangular decomposition of* $\mathcal{U}_q(\mathfrak{g})$*,*
(b) $\dim V^q_\lambda = 1$*,*
(c) $\dim V^q_\mu < \infty$ *for all* $\mu \in \mathrm{wt}(V^q)$*,*
(d) and $V^q = \bigoplus_{\mu \leq \lambda} V^q_\mu$*.*

1.22. **Definition.** Fix $\lambda \in P$. Then, we define

(a) $J^q(\lambda)$ to be the left $\mathcal{U}_q(\mathfrak{g})$ ideal generated e_i for $i \in I$ and $K_h - q^{\lambda(h)}1$ for $h \in P^\vee$ and
(b) the *Verma module* $M^q(\lambda) = \mathcal{U}_q(\mathfrak{g})/J^q(\lambda)$.

1.23. **Example.** For $\mathcal{U}_q(\mathfrak{sl}_2)$, we have that $J^q(1) = (e, K - q \cdot 1)$. Then, observe in $M^q(1) = \mathcal{U}_q(\mathfrak{sl}_2)/J^q(1)$ that

$$\begin{cases} e.(1 + J^q(1)) = e + J^q(1) = J^q(1) = 0 \\ K.(1 + J^q(1)) = K + J^q(1) = q \cdot (1 + J^q(1)) \end{cases}$$

so $1 + J^q(1)$ is a maximal weight vector of weight 1. Furthermore,

$$K.f.(1 + J^q(1)) = q^{-\alpha(h)} f K^{-1}(1 + J^q(1)) = q^{-1} f.(1 + J^q(1))$$

So, by iterating this computation, we see that $f^m.(1 + J^q(1))$ has weight $1 - 2m$.

1.24. **Proposition.** [HK02, Proposition 3.2.2]

(a) The Verma module $M^q(\lambda)$ is a highest weight module with highest weight λ and highest weight vector $v_\lambda = 1 + J^q(\lambda)$.

(b) As a $\mathcal{U}_q^-$-module, $M^q(\lambda)$ is free of rank 1 generated by the highest weight vector v_λ.

(c) Every highest weight $\mathcal{U}_q(\mathfrak{g})$-module with highest weight λ is a homomorphic image of $M^q(\lambda)$.

(d) The Verma module $M^q(\lambda)$ has a unique maximal submodule, denoted $N^q(\lambda)$.

(e) The quotient $M^q(\lambda)/N^q(\lambda)$ is an irreducible highest weight module with highest weight λ. We will denote this module by $V^q(\lambda)$.

1.25. **Definition.** (a) A weight module V^q over the quantum group $\mathcal{U}_q(\mathfrak{g})$ is called *integrable* if all e_i and f_i for $i \in I$ are locally nilpotent on V^q, that is, for each e_i, there exists a positive integer N such that $e_i^N v = 0$ for all $v \in V^q$, and similarly for the f_i.

(b) The *category* $\varnothing_{int}^q$ is the category consisting of all $\mathcal{U}_q(\mathfrak{g})$-modules in the category $\varnothing^q$ that are integrable.

(c) For a fixed $i \in I$, let the $\mathcal{U}_q(\mathfrak{g})$-subalgebra $\mathcal{U}_q(\mathfrak{g}_{(i)}) = \langle e_i, f_i, \tilde{K}_i^{\pm} \rangle$.

1.26. **Proposition.** *(a) $\mathcal{U}_q(\mathfrak{g}_{(i)}) \cong \mathcal{U}_{q_i}(\mathfrak{sl}_2)$*

(b) [HK02, Proposition 3.2.4] *Let $V^q \in \varnothing_{int}^q$. Then, for each $i \in I$, V^q decomposes into a direct sum of $\mathcal{U}_q(\mathfrak{h})$-invariant finite dimensional irreducible $\mathcal{U}_q(\mathfrak{g}_{(i)})$-submodules.*

(c) [HK02, Proposition 3.2.6] *If we let $f_i^{(k)} := f_i^k/[k]_{q_i}!$ be the divided power of f_i, then, for $\lambda \in P^+$ and v_λ the highest weight vector of $V^q(\lambda)$, we have*

$$f_i^{\lambda(h_i)+1} v_\lambda = 0 \text{ for all } i \in I$$

(d) [HK02, Proposition 3.2.7] *A highest weight $\mathcal{U}_q(\mathfrak{g})$-module V^q with highest weight $\lambda \in P$ and highest weight vector v_λ is integrable if and only if for every $i \in I$, there exists some N_i such that $f_i^{N_i} v_\lambda = 0$.*

(e) [HK02, Proposition 3.2.8] *Let $V^q(\lambda)$ be the irreducible highest weight $\mathcal{U}_q(\mathfrak{g})$-module with highest weight $\lambda \in P$. Then $V^q(\lambda) \in \varnothing_{int}^q$ if and only if $\lambda \in P^+$.*

1.3. $\mathbb{A}_1$-forms.

1.27. **Definition.** We define the following terminology.

(a) Let $\mathbb{A}_1$ be the localization of $F[q]$ at the ideal $(q-1)$ so that

$$\mathbb{A}_1 = \{f(q) \in F(q) \mid f \text{ is regular at } q = 1\} = \{g/h \mid g, h \in F[q], h(1) \neq 0\}$$

(b) Define, for $n \in \mathbb{Z}$,

$$[y; n]_x := \frac{yx^n - y^{-1}x^{-n}}{x - x^{-1}}$$

and

$$(y; n)_x := \frac{yx^n - 1}{x - 1}$$

Notice that $[n]_{q_i} \in \mathbb{A}_1$ and $\binom{m}{n}_{q_i} \in \mathbb{A}_1$ since they are elements of $\mathbb{Z}[q, q^{-1}]$.

1.28. **Example.** (a)

$$[q_i^m; n]_{q_i} = \frac{q_i^{m+n} - q_i^{-m-n}}{q_i - q_i^{-1}} = [m+n]_{q_i} \in \mathbb{A}_1$$

(b)

$$(q_i^m; n)_{q_i} = \frac{q_i^{m+n} - 1}{q_i - 1} \in \mathbb{A}_1$$

since we can factor $(q_i^m; n)_{q_i} = q_i^{m+n-1} + q_i^{m+n-2} + \cdots + 1$.

(c)

$$[K_h; n]_q = \frac{K_h q^n - K_{-h} q^{-n}}{q - q^{-1}} \in \mathcal{U}_q^0$$

(d)

$$(K_h; n)_q = \frac{K_h q^n - 1}{q - 1} \in \mathcal{U}_q^0$$

1.29. **Definition.** (a) Define the $\mathbb{A}_1$-*form*, denoted $\mathcal{U}_{\mathbb{A}_1}$, of the quantum group $\mathcal{U}_q(\mathfrak{g})$ to be the $\mathbb{A}_1$-subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by the elements e_i, f_i, K_h, and $(K_h; 0)_q$ for $i \in I, h \in P^\vee$.

(b) Let $\mathcal{U}_{\mathbb{A}_1}^+$ (resp $\mathcal{U}_{\mathbb{A}_1}^-$) be the $\mathbb{A}_1$-subalgebra of $\mathcal{U}_{\mathbb{A}_1}$ generated by the elements e_i (resp f_i) for $i \in I$.

(c) Let $\mathcal{U}_{\mathbb{A}_1}^0$ be that $\mathbb{A}_1$-subalgebra of $\mathcal{U}_{\mathbb{A}_1}$ generated by K_h and $(K_h; 0)_q$ for $h \in P^\vee$.

1.30. **Lemma.** [HK02, Lemma 3.3.2]

(a) $(K_h; n)_q \in \mathcal{U}_{\mathbb{A}_1}^0$ *for all* $n \in \mathbb{Z}$ *and* $h \in P^\vee$.

(b) $[K_i; n]_{q_i} \in \mathcal{U}_{\mathbb{A}_1}^0$ *for all* $n \in \mathbb{Z}$ *and* $i \in I$.

1.31. **Proposition.** [HK02, Proposition 3.3.3] *We have a natural isomorphism of* $\mathbb{A}_1$*-modules*

$$\mathcal{U}_{\mathbb{A}_1} \cong \mathcal{U}_{\mathbb{A}_1}^- \otimes \mathcal{U}_{\mathbb{A}_1}^0 \otimes \mathcal{U}_{\mathbb{A}_1}^+$$

induced from the triangular decomposition of $\mathcal{U}_q(\mathfrak{g})$.

1.32. **Definition.** The $\mathbb{A}_1$-form of the highest weight module V^q with highest weight $\lambda \in P$ and highest weight vector v_λ is defined to be the $\mathcal{U}_{\mathbb{A}_1}$-module $V_{\mathbb{A}_1} = \mathcal{U}_{\mathbb{A}_1} v_\lambda$.

1.33. **Proposition.** *Letting* V^q *be as above,* *(a)* [HK02, Proposition 3.3.5] $V_{\mathbb{A}_1} = \mathcal{U}^-_{\mathbb{A}_1} v_\lambda$

(b) [HK02, Proposition 3.3.6]

$$V_{\mathbb{A}_1} = \bigoplus_{\mu \leq \lambda} (V_{\mathbb{A}_1})_\mu \text{ where } (V_{\mathbb{A}_1})_\mu = V_{\mathbb{A}_1} \cap V^q_\mu$$

(c) [HK02, Proposition 3.3.7] *For each $\mu \in P$, the weight space $(V_{\mathbb{A}_1})_\mu$ is a free $\mathbb{A}_1$-module with* $\operatorname{rank}_{\mathbb{A}_1}(V_{\mathbb{A}_1})_\mu = \dim_{F(q)} V^q_\mu$.

(d) [HK02, Proposition 3.3.8] *The $F(q)$-linear map $\phi\colon F(q) \otimes_{\mathbb{A}_1} V_{\mathbb{A}_1} \to V^q$ given by $c \otimes v \mapsto cv$ is an isomorphism.*

1.34. **Remark.** [HK02, Remark 3.3.9]

(a) From the last part of the proposition, one can deduce that the $\mathbb{A}_1$-form $V_{\mathbb{A}_1}$ of a highest weight module V^q is an integral form over $\mathbb{A}_1$. In other words, it can be viewed as an $\mathbb{A}_1$-lattice in V^q.

(b) Item (c) can be generalized; see [HK02, Exercise 3.10].

1.4. **Classical Limit.** The classical limit construction demonstrates how the Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ really is a deformation of the Hopf algera structure of $\mathcal{U}(\mathfrak{g})$ and how you can recover the structure by taking $q \to 1$. Furthermore, this limit will allow us to see that various quantum group representations truly correspond to their classical Lie theoretic counterparts. Most notably, the characters remain unchanged by the limit construction and this allows us to prove uniquenss results about the irreducible quantum group representations.

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