INTRODUCTION TO THE STRUCTURE OF SEMISIMPLE LIE ALGEBRAS AND THEIR REPRESENTATION THEORY

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1. Introduction

The representation theory of semisimple Lie algebras is an incredibly beautiful model of a representation theory, since the simple representations of semisimple Lie algebras are completely classified. Furthermore, such a tool is very powerful, since it can be used to

- completely classify all simple Lie algebras
- completely understand the representation theory of simply connected Lie groups
- serve as a guide for understanding the reprsentation theory of other algebraic objects

In this monograph, we will be more concerned with the former application and then touch on the general representation theory of semisimple Lie algebras.

Note that, in this monograph, a *linear Lie algebra* is a Lie algebra that is viewed a subalgebra of $\mathfrak{gl}(V)$, although Ado's theorem shows that any Lie algebra can be realized in such a way. Furthermore, a field F will always be of characteristic 0 and algebraically closed unless otherwise stated.

2. Weyl's Theorem

In this section, we seek to prove Weyl's theorem, which provides us with a characterization of the structure of representations of semisimple Lie algebras. This section is mostly a rewriting and expanding of [Hum72] chapter 6.

2.1. **Theorem** (Weyl's Theorem). *Let $\mathfrak{g}$ be a semisimple Lie algebra. If $\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$, then ϕ is completely reducible.*

To prove this theorem, we will need to show that, for any $\mathfrak{g}$-module V and any submodule $W \leq V$, we get $V = W \oplus W'$, where W' is a complement of W. The easiest way to do this will be to find some module homomorphism

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$f\colon V \to W$ with $\ker f \cap W = \{0\}$ and $W + \ker f = V$. To construct such a homomorphism, we will need the following.

2.2. **Definition.** Let $\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful representation of a Lie algebra $\mathfrak{g}$ and let $\beta(x,y) := \operatorname{tr}(\phi(x)\phi(y))$ be a bilinear form on $\mathfrak{g}$. Then, for basis $\{x_1,\dots,x_n\}$ of $\mathfrak{g}$ and $\{y_1,\dots,y_n\}$ of $\mathfrak{g}$ such that $\beta(x_i,y_j) = \delta_{ij}$, we define the *Casimir element of ϕ* to be

$$c_\phi(\beta) = \sum_i \phi(x_i)\phi(y_i) \in \operatorname{End}(V)$$

2.3. **Example.** Let $\mathfrak{g} = \mathfrak{sl}_2$ with standard basis $\{e,h,f\}$. Then, as shown in [See18], we have dual basis $\{\frac{1}{4}f, \frac{1}{8}h, \frac{1}{4}e\}$. Now, if ϕ is the adjoint representation, we get

$$\begin{aligned}
c_\phi(\{e,f,h\}) &= \phi(e)\phi(\frac{1}{4}f) + \phi(h)\phi(\frac{1}{8}h) + \phi(f)\phi(\frac{1}{4}e)\\
&= \frac{1}{4}\operatorname{ad}_e \operatorname{ad}_f + \frac{1}{8}\operatorname{ad}_h^2 + \frac{1}{4}\operatorname{ad} f \operatorname{ad}_e\\
&= \frac{1}{4}(2e_{1,1} + 2e_{2,2}) + \frac{1}{8}(4e_{1,1} + 4e_{3,3}) + \frac{1}{4}(2e_{2,2} + 2e_{3,3})\\
&= e_{1,1} + e_{2,2} + e_{3,3}\\
&= Id_3
\end{aligned}$$

2.4. **Lemma.** *Take a basis $\{x_1,\dots,x_n\}$ of $\mathfrak{g}$ and a basis $\{y_1,\dots,y_n\}$ of $\mathfrak{g}$ such that, for nondegenerate symmetric associative bilinear form β on $\mathfrak{g}$, $\beta(x_i,y_j) = \delta_{ij}$. For $x \in \mathfrak{g}$, if*

$$[x,x_i] = \sum_j a_{ij}x_j, \quad [x,y_i] = \sum_j b_{ij}y_j$$

then $a_{ik} = -b_{ki}$

Proof. We compute

$$\begin{aligned}
a_{ik} &= \sum_j a_{ij}\delta_{jk}\\
&= \sum_j a_{ij}\beta(x_j,y_k) && \text{by bilinearity}\\
&= \beta([x,x_i],y_k)\\
&= \beta(-[x_i,x],y_k)\\
&= \beta(x_i,-[x,y_k]) && \text{by associativity}\\
&= -\sum_j b_{kj}\beta(x_i,y_j)\\
&= -b_{ki}
\end{aligned}$$

□

2.5. **Example.** Continuing the example above, consider basis $\{e, h, f\}$ of $\mathfrak{sl}_2$ and dual basis with respect to the Killing form, $\{\frac{1}{4}f, \frac{1}{8}h, \frac{1}{4}e\}$. Then, for $x = ae + bh + cf$, we check that

$$\begin{aligned}
[x, e] &= 2be - ch & &\implies a_{1,1} = 2b, a_{1,2} = -c, a_{1,3} = 0\\
[x, h] &= -2ae + 2cf & &\implies a_{2,1} = -2a, a_{2,2} = 0, a_{2,3} = 2c\\
[x, f] &= ah - 2bf & &\implies a_{3,1} = 0, a_{3,2} = a, a_{3,3} = -2b\\
[x, \frac{1}{4}f] &= -\frac{1}{2}bf + \frac{1}{4}ah & &\implies b_{1,1} = -2b, b_{1,2} = 2a, b_{1,3} = 0\\
[x, \frac{1}{8}h] &= \frac{1}{4}cf - \frac{1}{4}ae & &\implies b_{2,1} = c, b_{2,2} = 0, b_{2,3} = -a\\
[x, \frac{1}{4}e] &= -\frac{1}{4}ch + \frac{1}{2}be & &\implies b_{3,1} = 0, b_{3,2} = -2c, b_{3,3} = 2b
\end{aligned}$$

2.6. **Lemma.** *For $x, y, z \in \mathrm{End}(V)$,*

$$[x, yz] = [x, y]z + y[x, z]$$

Proof.

$$\begin{aligned}
[x, y]z + y[x, z] &= (xy - yx)z + y(xz - zx)\\
&= xyz - yxz + yxz - yzx\\
&= xyz - yzx\\
&= [x, yz]
\end{aligned}$$

□

2.7. **Proposition.** *$c_\phi(\beta)$ commutes with $\phi(\mathfrak{g})$ in* $\mathrm{End}(V)$.

Proof. For $x \in \mathfrak{g}$, consider $[\phi(x), c_\phi(\beta)]$. Using the lemma immediately above, we get

$$[\phi(x), c_\phi(\beta)] = \sum_i [\phi(x), \phi(x_i)\phi(y_i)] = \sum_i [\phi(x), \phi(x_i)]\phi(y_i) + \phi(x_i)[\phi(x), \phi(y_i)]$$

Moreover, we know that $[\phi(x), \phi(x_i)] = \sum_j a_{ij}\phi(x_j)$ and $[\phi(x), \phi(y_i)] = \sum_j b_{ij}\phi(x_j)$, with $a_{ik} = -b_{ki}$ for any k using the same decomposition as in the first of the two lemmas. Thus,

$$[\phi(x), c_\phi(\beta)] = \sum_{i,j} a_{ij}\phi(x_j)\phi(y_i) + \sum_{i,j} b_{i,j}\phi(x_i)\phi(y_j) = 0$$

□

This proposition is vitally important for proving Weyl's theorem. With this element, we now have an object that must "behave like a scalar" with respect to $\phi(\mathfrak{g})$ in $\mathrm{End}(V)$. Indeed, in the $\mathfrak{sl}_2$ example, our Casimir element is exactly the identity, although this need not always be true:

2.8. **Proposition.** *If ϕ is faithful and irreducible, then $c_\phi(\beta)$ is a scalar in* $\mathrm{End}(V)$.

Proof. Since ϕ is irreducible, we can invoke Schur's Lemma to get that $c_\phi(\beta)$ must be a scalar multiple of the identity map. $\square$

2.9. **Remark.** Note that $c_\phi(\beta)$ is independent of choice of basis when ϕ is faithful, so we will often denote $c_\phi := c_\phi(\beta)$. For more details, see [Hum72, p 27].

2.10. **Proposition.** *Given Casimir element c_ϕ of faithful representation $\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$, we get that*

$$\operatorname{tr}(c_\phi) = \dim \mathfrak{g}$$

Proof. This follows from simply applying the trace operator.

$$\operatorname{tr}(c_\phi) = \sum_i \operatorname{tr}(\phi(x_i)\phi(y_i)) = \sum_i \beta(x_i, y_i) = \dim \mathfrak{g}$$

$\square$

2.11. **Example.** The Casimir element of $\mathfrak{sl}_2$ under the adjoint representation was precisely Id_3 and $\operatorname{tr}(Id_3) = 3 = \dim \mathfrak{sl}_2$.

2.12. **Lemma.** *Let $\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a semisimple Lie algebra $\mathfrak{g}$. Then, $\phi(\mathfrak{g}) \subseteq \mathfrak{sl}(V)$.*

Proof. We know that, for a semisimple Lie algebra, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ by the fact that $\mathfrak{g}$ is a direct sum of simple Lie algebras. Furthermore, $[\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$ by simple computation. Thus, $\phi(\mathfrak{g}) = [\phi(\mathfrak{g}), \phi(\mathfrak{g})] \subseteq [\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$. $\square$

2.13. **Lemma.** *Let $\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a semisimple Lie algebra and let $W \leq V$ be irreducible with codimension 1. Then, ϕ is completely reducible.*

Proof. Without loss of generality, we may assume ϕ is faithful . Now, consider c_ϕ commutes with $\phi(\mathfrak{g})$, so $c_\phi(g.v) = (c_\phi \circ \phi(g))(v) = (\phi(g) \circ c_\phi)(v)$ tells us that c_ϕ is actually a $\mathfrak{g}$-module hormormophism on V. Thus, $c_\phi(W) \subseteq W$ and $\ker c_\phi$ is a submodule of V. Now, by the lemma above, $\mathfrak{g}$ must send all of V into W and thus act trivially on the quotient module V/W. Therefore, c_ϕ must also act trivially since it is simply a linear combination of actions in $\mathfrak{g}$. So, it must be that the trace of c_ϕ on V/W is 0. However, we know $\operatorname{tr}_V(c) = \dim \mathfrak{g} \neq 0$. Now, since c_ϕ acts as a (must be non-zero) scalar on W, it must be that $\ker c_\phi \cap W = \{0\}$ and $\ker c_\phi + W = V$. Thus, since $\ker c_\phi$ is a submodule of V, it must be that $\ker c_\phi$ is 1-dimensional and $V = W \oplus \ker c_\phi$. $\square$

why?

2.14. **Lemma.** *Let $\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a semisimple Lie algebra and let $W \leq V$ with codimension 1. Then, ϕ is completely reducible.*

Proof. Note that this is the same as the lemma above, but now W need not be irreducible. We will show that we can reduce this lemma to the case of the previous lemma 2.13. By 2.12, we know $\mathfrak{g}$ must act trivially on V/W. Thus, $V/W \cong F$ and we have short exact sequence

$$0 \to W \to V \to F \to 0$$

Now, if $\dim W = 1$, W is irreducible and we can appeal to the previous lemma. So, now we proceed by induction and assume the result it true for $\dim W < n$. Let $\dim W = n$ and W have proper submodule W'. Then, we have short exact sequence

$$0 \to W/W' \to V/W' \to F \to 0$$

Now, by induction, $V/W' = W/W' \oplus W''/W'$. However, this lifts to a short exact sequence

$$0 \to W' \to W'' \to F \to 0$$

However, $\dim W' < \dim W \implies \dim W'' = \dim W' + 1 < \dim W + 1 = \dim V$, so we can apply induction to get a one dimensional submodule complementary to W' in W'', say X, so that $W'' = W' \oplus X$. Thus, we get

$$V/W' = W/W' \oplus W''/W' = W/W' \oplus (W' \oplus X)/W' \implies V = W \oplus X$$

since $W \cap X = 0$ and $\dim W + \dim X = \dim V$. Thus, we can repeat this process until we get an irreducible W and appeal to the lemma above (2.13) to get the result. $\square$

Proof of Weyl's Theorem 2.1. Let $0 \neq W \subsetneq V$ be $\mathfrak{g}$-modules. Thus, we have short exact sequence

$$0 \to W \to V \to V/W \to 0$$

Let $U = \{\psi \in \mathrm{Hom}(V, W) \mid \exists c \in F, \psi|_W(w) = cw, \forall w \in W\}$ and let $T = \{\psi \in \mathrm{Hom}(V, W) \mid \psi|_W(w) = 0, \forall w \in W\} \leq U$. Then, we check that U and T are $\mathfrak{g}$-submodules of $\mathrm{Hom}(V, W)$. For $g \in \mathfrak{g}$, $\psi \in U$, and $w \in W$

$$\begin{aligned}
(g.\psi)(w) &= g.\psi(w) - \psi(g.w) && \text{by the Jacobi identity} \\
&= g.(cw) - c(g.w) && \psi \in U \\
&= c(g.w) - c(g.w) \\
&= 0
\end{aligned}$$

Thus, $g.\psi|_W = 0$ for all $g \in \mathfrak{g}$. A similar computation shows that T is also a $\mathfrak{g}$-sucmodule, specifying $c = 0$. This also shows that $\mathfrak{g}.U \leq T$. Now, consider the quotient U/T. Every element in U/T is uniquely determined by the scalar c, that is $\tau \in \psi + T$ if and only if $\tau|_W = \psi|_W = c.1_W$ for scalar c. Thus, we have short exact sequence

$$0 \to T \to U \to F \to 0$$

and thus, by the lemma above (2.14), $U = T \oplus T'$ where T' is a one-dimensional $\mathfrak{g}$-submodue. If we let $T' = (\psi)$, $\psi \in \mathrm{End}(V, W)$, then we can scale ψ such that $\psi|_W = 1_W$ since $\psi \notin T$. Now, since ψ is a $\mathfrak{g}$-module homomorphism, $\ker \psi \leq V$, but $\ker \psi \neq V$ since $W \cap \ker \psi = \{0\}$. However, ψ cannot be injective because $W \subsetneq V$, which is finite-dimensional, and thus

$\ker\psi \neq \{0\}$. Therefore, since ψ is surjective, we have that $V = W \oplus \ker\psi$ as a non-trivial decomposition of V. Thus, we can iteratively continue this process until V has been completely reduced. $\square$

Weyl's theorem has many applications in the representation theory of semisimple Lie algebras. One immediate consequence is the preservation of the Jordan decomposition ([Hum72] section 6.4). That is,

2.15. **Theorem.** *Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a semisimple linear Lie algebra where V is finite-dimensional. Then, $\mathfrak{g}$ contains the semisimple and nilpotent parts in $\mathfrak{gl}(V)$ of all its elements. In particular, the abstract and usual Jordan decompositions in $\mathfrak{g}$ coincide.*

2.16. **Corollary.** *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a (finite-dimensional) representation of $\mathfrak{g}$. If $x = s+n$ is the abstract Jordan decomposition of $x \in \mathfrak{g}$, then $\phi(x) = \phi(s)+\phi(n)$ is the usual Jordan decomposition of $\phi(x)$.*

2.17. **Remark.** Note that this theorem is a generalization of the following facts.

- A nilpotent $x \in \mathfrak{g}$ is also ad-nilpotent.
- If $x \in \mathfrak{g}$ is diagonalizable, then ad_x is also diagonailzable.

We also prove the following, which does not require the theorems above, but is related to the Jordan decomposition.

2.18. **Proposition.** *Let $\mathfrak{g}$ be a semisimple Lie algebra, thus decomposing into simple ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_t$. Then, the semisimple and nilpotent parts of $x \in \mathfrak{g}$ are the sums of the semisimple and nilpotent parts in the various $\mathfrak{g}_i$ of the components of x.*

Proof. Take $x \in \mathfrak{g}$, so $x = x_1 + \cdots + x_t$ with $x_i \in \mathfrak{g}_i$ and $x_i = (x_i)_s + (x_i)_n$. Then, $(\mathrm{ad}_{\mathfrak{g}}(x_i)_s)|_{\mathfrak{g}_i} = \mathrm{ad}_{\mathfrak{g}_i}(x_i)_s$ is a semisimple endomorphism of $\mathfrak{g}_i$ and $(\mathrm{ad}_{\mathfrak{g}}(x_i)_s)|_{\mathfrak{g}_j} = 0$ for $i \neq j$ by the direct sum decomposition. Thus, $\mathrm{ad}_{\mathfrak{g}}(x_i)_s$ is a semisimple endomorphism of $\mathfrak{g}$. Now, let $u = (x_1)_s + \cdots + (x_t)_s$. Then, $\mathrm{ad}_{\mathfrak{g}}\, u$ is a semisimple endomorphism of $\mathfrak{g}$ since $[(x_i)_s, (x_j)_s] = 0$ for all $i \neq j$.

By a similar argument, if $v = (x_1)_n + \cdots + (x_t)_n$, then $\mathrm{ad}_{\mathfrak{g}}\, v$ is a nilpotent endomorphism of $\mathfrak{g}$. Therefore, because

$$\begin{aligned}
[u, v] &= [(x_1)_s, (x_1)_n] + [(x_1)_s, (x_2)_n] + [(x_2)_n, (x_1)_s] + [(x_2)_s, (x_2)_n] + \cdots \\
&= [(x_1)_s, (x_1)_n] + \cdots + [(x_t)_s, (x_t)_n] \\
&= 0
\end{aligned}$$

Then u, v are commuting endomorphisms with u semisimple and v nilpotent, and so by the uniqueness of Jordan decomposition, $x = u + v$ is the Jordan decomposition of x. $\square$

3. Representations of $\mathfrak{sl}_2(F)$

Recall that $\mathfrak{sl}_2(F)$ (where F is an algebraically closed field) is the Lie algebra of 2 by 2 traceless matrices with standard basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and relations

$$[h, e] = 2e, \ [h, f] = -2f, \ [e, f] = h$$

3.1. **Proposition.** *Let V be an arbitrary $\mathfrak{sl}_2(F)$-module. Then, h acts diagonally on V, that is, $\phi(h)$ is diagonalizable.*

Proof. Since h is semisimple, we get that $\phi(h)$ is semisimple by the corollary above. $\square$

3.2. **Proposition.** *Let V be an arbitrary $\mathfrak{sl}_2(F)$-module. Then, V decomposes as a direct sum of eigenspaces.*

$$V = \bigoplus_{\lambda \in F} V_\lambda, \ V_\lambda = \{v \in V \mid h.v = \lambda v\}$$

Proof. Since h is semisimple, $\phi(h)$ is semisimple and thus diagonalizable, meaning that the set of eigenvectors of $\phi(h)$ forms a basis for V. Thus, we can decompose V into $\phi(h)$-eigenspaces. $\square$

3.3. **Definition.** Whenever $V_\lambda \neq 0$, we call λ a *weight* of h in V, and we call V_λ a *weight space*.

3.4. **Lemma.** *Let V be a finite-dimensional $\mathfrak{g}$-module. Then, for $v \in V_\lambda$, $e.v \in V_{\lambda+2}$ and $f.v \in V_{\lambda-2}$.*

Proof. This follows from the following computations:

$$\begin{aligned} h.(e.v) &= [h, e].v + e.h.v && \text{by the Jacobi identity} \\ &= 2e.v + e.(\lambda v) && \text{by } \mathfrak{sl}_2 \text{ relations} \\ &= (\lambda + 2)e.v \end{aligned}$$

Similarly, we get

$$h.(f.v) = [h, f].v + f.h.v = -2f.v + f.(\lambda v) = (\lambda - 2)f.v$$

$\square$

3.5. **Proposition.** *For finite-dimensional $\mathfrak{sl}_2$-module V, there exists a λ such that $V_\lambda \neq 0$ but $V_{\lambda+2} = 0$.*

Proof. Since V is finite-dimensional and $V = \bigsqcup_{\lambda \in F} V_\lambda$ is direct, such a λ must exist. $\square$

3.6. **Definition.** A nonzero vector in V_λ annihilated by $e \in \mathfrak{sl}_2$ is called a *maximal vector of weight λ* or the *highest weight vector of weight λ*.

3.7. **Lemma.** *Let V be an irreducible $\mathfrak{sl}_2(F)$-module. Let v_0 be a maximal vector of weight λ and set $v_{-1} = 0$ and $v_k = \frac{1}{k!} f^k.v_0$ for $k \geq 0$. Then,*

(a) $h.v_k = (\lambda - 2k)v_k$
(b) $f.v_k = (k+1)v_{k+1}$
(c) $e.v_k = (\lambda - k + 1)v_{k-1}$

Proof. By the previous lemma, $v_k \in V_{\lambda - 2k}$, so $h.v_k = (\lambda - 2k)v_k$.

By definition of v_k,

$$f.v_k = f.\frac{1}{k!} f^k.v_0 = \frac{1}{k!} f^{k+1}.v_0 = (k+1) f^{k+1}.v_0 = (k+1)v_{k+1}$$

For the final relation, we use induction on k. When $k = 0$, we have $e.v_0 = 0$ since v_0 is a highest weight vector, and thus our formula holds since $v_{-1} = 0$ by assumption. Now, observe

$$\begin{aligned}
ke.v_k &= e.f.v_{k-1} && \text{by part (b), } f.v_j = (j+1)v_{j+1} \\
&= [e, f].v_{k-1} + f.e.v_{k-1} && \text{by the Jacobi identity} \\
&= h.v_{k-1} + f.e.v_{k-1} && \text{by } \mathfrak{sl}_2 \text{ relations} \\
&= (\lambda - 2(k-1))v_{k-1} + (\lambda - k + 2)f.v_{k-2} && \text{by part (a) and induction} \\
&= (\lambda - 2k + 2)v_{k-1} + (k-1)(\lambda - k + 2)v_{k-1} && \text{by part (b)} \\
&= k(\lambda - k + 1)v_{k-1}
\end{aligned}$$

Thus, dividing both sides by k, we get the desired result. □

3.8. **Proposition.** *Let V be an irreducible $\mathfrak{sl}_2(F)$-module. Then, for m the smallest integer such that $v_m \neq 0, v_{m+1} = 0$, we get basis $\{v_0, v_1, \ldots, v_m\}$ for V.*

Proof. From formula (a) in the lemma above, it is clear that $\{v_0, \ldots, v_m\}$ are linearly independent since they are eigenvectors of different eigenvalues for the h-action. Furthermore, the formulas in the lemma above show us that $\text{span}\{v_0, v_1, \ldots, v_m\}$ is a non-zero submodule of V. However, since V is irreducible, that must mean that V is the span of this linearly independent set. □

3.9. **Proposition.** *Let V be an irreducible $\mathfrak{sl}_2(F)$-module. Then, given highest weight vector v of weight λ, λ must be an integer.*

Proof. In the lemma above, take $k = m + 1$ for formula (c). We get, for $v_m \neq 0$ but $v_{m+1} = 0$,

$$0 = e.v_{m+1} = (\lambda - (m+1) + 1)v_m = (\lambda - m)v_m$$

Thus, it must be that $\lambda = m \in \mathbb{Z}$. □

3.10. **Definition.** We call the weight of a maximal vector $v_m \in V$, an irreducible $\mathfrak{sl}_2(F)$-module, the *highest weight* of V.

3.11. **Proposition.** *Let m be the highest weight for irreducible $\mathfrak{sl}_2(F)$-module V. Then,* $\dim V = m+1$ *and* $\dim V_\mu = 1$ *if* $V_\mu \neq 0$.

Proof. Given the eigenbasis (relative to the h-action) $\{v_0, \ldots, v_m\}$, it is immediate that $\dim V = m+1$. Furthermore, for each $V_\mu \neq 0$, there is a unique eigenvector under the h-action spanning it, up to scalar multiple. Thus, $\dim V_\mu = 1$. □

To summarize, we have the following theorem.

3.12. **Theorem.** *Let V be an irreduclbe $\mathfrak{sl}_2(F)$-module.*

(a) *Relative to h, V is a direct sum of weight spaces V_μ where $\mu = m, m-2, \ldots, -(m-2), -m$ where $m+1 = \dim V$ and $\dim V_\mu = 1$ for each μ.*
(b) *V has (up to nonzero scalar multiples) a unique maximal vector, whose weight is m.*
(c) *The action of $\mathfrak{sl}_2(F)$ on V is given explicitly by the formulas in the lemma above, if the basis is chosen appropriately. In particular, there exists at most one irreducible $\mathfrak{sl}_2(F)$-module (up to isomorphism) of each possible dimension $m+1$ where $m \in \mathbb{Z}, m \geq 0$.*

Proof. Part (a) is 3.11. Since $\dim V = m+1$, we see by relations that V has a highest weight vector of weight m, say v_m, and it is unique since its eigenspace, V_μ, is 1-dimensional. Finally, part (c) is just appealing to the lemma above. □

3.13. **Corollary.** *Let V be any (finite-dimensional) $\mathfrak{sl}_2(F)$-module. Then, the eigenvalues of h on V are all integers, and each occurs along with its negative (an equal number of times). Moreover, in any description of V into a direct sum of irreducible submodules, the number of summands is precisely* $\dim V_0 + \dim V_1$.

Proof. If $V = 0$, then the result is immediate. So, given a non-trivial V, Weyl's theorem says V is completely reducible, so we write V as a direct sum of irreducible submodules. Now, each of these irreducible summands must have integer eigenvalues of h and thus, so must V. For the second part of the corollary, we note that an irreducible $\mathfrak{sl}_2$-module must have an occurence of a 0-eigenvector or a 1-eigenvector, but it cannot have both since the eigenvalues differ by even numbers due to the fact that $h.v_k = (\lambda - 2k)v_k$ in the lemma above. □

Let us now explicitly describe these representations, which are easy to enumerate due to the theorem above.

3.14. **Example.** The easiest representation is the trivial one-dimensional representation $\mathbb{C} = V_0$.

3.15. **Example.** Consider $V = \{e_1, e_2\}$, the "standard representation" of $\mathfrak{sl}_2$, where $h.e_1 = e_1$ and $h.e_2 = -e_2$. Then, $V = V_{-1} \oplus V_1 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. By the corollary above, this representation must be irreducible.

3.16. **Example.** Consider $W = \mathrm{Sym}^2 V$ for V the standard representation above. Then, a basis is given by $\{e^2, ef, f^2\}$ and we have relations

$$\begin{aligned} h.(e \cdot e) &= e \cdot h.e + h.e \cdot e = 2e^2 \\ h.(e \cdot f) &= e \cdot h.f + h.e \cdot f = 0 \\ h.(f \cdot f) &= f \cdot h.f + h.f \cdot f = -2f^2 \end{aligned}$$

and so $W = \mathbb{C}f^2 \oplus \mathbb{C}ef \oplus \mathbb{C}e^2 = V_{-2} \oplus V_0 \oplus V_2$. Thus, by the corollary above, W is the irreducible representation of dimension 3. Proceeding more generally, one can conclude that the irreducible $\mathfrak{sl}_2$-representation of dimension n is given by $\mathrm{Sym}^n V$ for V the standard representation by computing

$$h.(e^{n-k} f^k) = (n-k) \cdot h.e \cdot e^{n-k-1} f^k + k \cdot h.f \cdot e^{n-k} f^{k-1} = (n-2k) \cdot e^{n-k} f^k$$

4. Representations of $\mathfrak{sl}_3(F)$

In this section, we describe the representations of $\mathfrak{sl}_3(F)$ following the program in [FH91]. In $\mathfrak{sl}_3(F)$, we do not have a single obvious element to give us an eigenspace decomposition, in contrast of $\mathfrak{sl}_2(F)$ where the single element h sufficed. Thus, instead, we will consider a 2-dimensional subspace of all diagonal matrices in $\mathfrak{sl}_3(F)$, denoted $\mathfrak{h} \subseteq \mathfrak{sl}_3(F)$. We make use of the fact

4.1. **Lemma.** *Commuting diagonalizable matrices are simultaneously diagonalizable.*

Therefore, we will be able to find simultaneous eigenvectors for the matrices in $\mathfrak{h}$ that yield an eigenspace decomposition with eigenvalues $\alpha \in \mathfrak{h}^*$.

4.2. **Proposition.** *Let V be an arbitrary $\mathfrak{sl}_3(F)$-module, then V decomposes as a direct sum of eigenspaces*

$$V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha, \;\; V_\alpha = \{v \in V \mid h.v = \alpha(h)v, \forall h \in \mathfrak{h}\}$$

Thus, looking specifically at $\mathfrak{sl}_3(F)$, we get

4.3. **Corollary.**

$$\mathfrak{sl}_3(F) \cong \mathfrak{h} \oplus \left(\bigoplus_\alpha \mathfrak{g}_\alpha \right)$$

where α ranges over a finite subset of $\mathfrak{h}^*$ given by $\{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq 3, i \neq j\}$ where

$$\epsilon_i \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_i$$

and, for any $h \in \mathfrak{h}, y \in \mathfrak{g}_\alpha$, we get $[h, y] = \alpha(h)y$.

Proof. The decomposition follows from the proposition and the finite-dimensionality of $\mathfrak{sl}_3(F)$. To see the specific decomposition, let $D \in \mathfrak{sl}_3(F)$ be a diagonal matrix with trace 0 and let $(m_{i,j}) = M \in \mathfrak{sl}_3(F)$. Then, by straightforward computation,

$$[D, M] = ((a_i - a_j)m_{i,j})$$

and so $[D, M]$ will be a multiple of M if and only if M has all but one entry equal to 0. Thus, $E_{i,j}$ generate the eigenspaces of the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}_3(F)$, given by the 6 linear functionals $\epsilon_i - \epsilon_j$. □

Now, given an $x \in \mathfrak{g}_\alpha$, we ask, how does its action affect this decomposition. We get the following.

4.4. **Proposition.** *Given $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$, $[x, y] \in \mathfrak{g}_{\alpha+\beta}$. In other words,* $\mathrm{ad}(\mathfrak{g}_\alpha)$ *sends vectors in $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\alpha+\beta}$.*

Proof. Consider the action of $\mathfrak{h}$ on $[x, y]$.

$$\begin{aligned}
[h, [x, y]] &= [x, [h, y]] + [[h, x], y] && \text{by the Jacobi identity} \\
&= [x, \beta(h)y] + [\alpha(h)x, y] && \text{since } y \in \mathfrak{g}_\beta, x \in \mathfrak{g}_\alpha \\
&= (\alpha(h) + \beta(h))[x, y]
\end{aligned}$$

□

Later, we will discuss how to visualize this information, but for now we are content with this understanding. More generally, we see that

4.5. **Proposition.** *Given a representation V of $\mathfrak{sl}_3$ with eigenspace decomposition $V = \bigoplus_\alpha V_\alpha$, then for any $x \in \mathfrak{g}_\alpha$ and $v \in V_\beta$, $x.v \in V_{\alpha+\beta}$. In other words, $\mathfrak{g}_\alpha$ takes V_β to $V_{\alpha+\beta}$.*

From the above situation, we also notice the following

4.6. **Proposition.** *The eigenvalues occurring in an irreducible representation of $\mathfrak{sl}_3(F)$ differ by integral linear combinations of the vectors $\{\epsilon_i - \epsilon_j\} \subseteq \mathfrak{h}^*$.*

Proof. Consider

$$\begin{aligned}
h.x.v &= x.h.v + [h, x].v && \text{by the Jacobi identity} \\
&= x.\beta(h)v + \alpha(h)x.v && \text{since } v \in V_\beta, x \in \mathfrak{g}_\alpha \\
&= (\alpha(h) + \beta(h))x.v
\end{aligned}$$

□

Thus, the $\epsilon_i - \epsilon_j$'s generate a *lattice* of eigenvalues in $\mathfrak{h}^*$.

4.7. **Definition.** The weights of the adjoint representation will be called *roots*, denoted Φ (this will be expanded on later) and the lattice generated by these roots is the *root lattice*, denoted Λ_R. Furthermore, from the decomposition $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$ in 4.3, the non-zero $\mathfrak{g}_\alpha$'s will be called *root spaces.*

Now, just as with $\mathfrak{sl}_2(F)$, we wish to find an equivalent notion to a highest weight vector for a representation of $\mathfrak{sl}_3(F)$.

4.8. **Lemma.** *Given a representation V of $\mathfrak{sl}_3(F)$, there exists a $v \in V$ such that*

(a) v is an eigenvector for $\mathfrak{h}$
(b) v is in the kernel of the action by $E_{1,2}, E_{1,3}$, and $E_{2,3}$ in $\mathfrak{sl}_3(F)$. (More generally, v is in the kernel of the action of half of the root spaces.)

Proof. Let us choose linear functional $\ell\colon \mathfrak{h}^* \to F$ given by

$$\ell(a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3) = aa_1 + ba_2 + ca_3$$

with $a + b + c = 0$ and $a > b > c$, such that $\ell|_{\Lambda_R}$ has trivial kernel. Thus, we get the equality

$$\{\mathfrak{g}_\alpha \subseteq \mathfrak{g} \mid \ell(\alpha) > 0\} = \{\mathfrak{g}_{\epsilon_1-\epsilon_3}, \mathfrak{g}_{\epsilon_2-\epsilon_3}, \mathfrak{g}_{\epsilon_1-\epsilon_2}\}$$

each of which corresponds to a basis matrix with a single nonzero entry above the diagonal. Now, pick a $v \in V_\alpha$ for $\ell(\alpha)$ maximal among the weights of V. Then, each of $E_{1,2}, E_{1,3}$, and $E_{2,3}$ must kill v by the maximality of α among the weights with respect to ℓ. □

4.9. **Definition.** A vector $v \in V$ that meets the properties of the lemma above is a *highest weight vector*.

4.10. **Proposition.** *Let V be an irreducible $\mathfrak{sl}_3(F)$ representation and $v \in V$ be a highest weight vector. Then, $V = \langle E_{2,1}, E_{3,1}, E_{3,2}\rangle.v$, that is, V is generated by v under successive application of $E_{2,1}, E_{3,1}$, and $E_{3,2}$.*

Proof. We show that the subspace $W \subseteq V$ spanned by images of v under successive application of $E_{2,1}, E_{3,1}$, and $E_{3,2}$ is preserved by all of $\mathfrak{sl}_3(F)$. Thus, we check

$$\begin{aligned}
E_{1,2}.v = E_{2,3}.v &= E_{1,3}.v = 0 && \text{by the lemma above}\\
E_{1,2}.E_{2,1}.v &= E_{2,1}.E_{1,2}.v + [E_{1,2}, E_{2,1}].v\\
&= 0 + \alpha([E_{1,2}, E_{2,1}])v && \text{since } [E_{1,2}, E_{2,1}] \text{ is a diagonal matrix} (\in \mathfrak{h})\\
E_{2,3}.E_{2,1}.v &= E_{2,1}.E_{2,3}.v + [E_{2,3}, E_{2,1}].v\\
&= 0 + 0 && \text{since } [E_{2,3}, E_{2,1}] = 0
\end{aligned}$$

Similar computations hold when applied to $E_{3,2}.v$. Furthermore, since $E_{1,3} = [E_{1,2}, E_{3,2}]$, we do not need to check it.

Now, let $W_n = \operatorname{span}\{w.v \mid w = w_n \cdots w_1, w_i \in \{E_{2,1}, E_{2,3}\}\}$ (remember that $E_{3,1} = [E_{3,2}, E_{2,1}]$). Then, $W = \bigcup_n W_n$. Now, consider that

$$\begin{aligned}
E_{1,2}.E_{2,1}.(w_{n-1} \cdots w_1).v &= E_{2,1}.E_{1,2}.(w_{n-1} \cdots w_1).v + [E_{1,2}, E_{2,1}].(w_{n-1} \cdots w_1).v\\
&\in E_{2,1}.W_{n-2} + \beta([E_{1,2}, E_{2,1}]).(w_{n-1} \cdots w_1).v && ([E_{1,2}, E_{2,1}] \in \mathfrak{h})
\end{aligned}$$

$$
\begin{aligned}
&\subseteq W_{n-1} \\
E_{2,3}.E_{2,1}.(w_{n-1}\cdots w_1).v &= E_{2,1}.E_{2,3}.(w_{n-1}\cdots w_1).v + [E_{2,3}, E_{2,1}].(w_{n-1}\cdots w_1).v \\
&\in E_{2,1}.W_{n-2} && ([E_{2,3}, E_{2,1}] = 0) \\
&\subseteq W_{n-1}
\end{aligned}
$$

and similarly for applying $E_{1,2}$ and $E_{2,3}$ to $E_{3,2}.(w_{n-1}\cdots w_1).v$. Thus, W is closed under the action $\mathfrak{sl}_3(F)$, but V is irreducible, so $W = V$. $\square$

In fact, we have slightly more.

4.11. **Proposition.** *Let V be a $\mathfrak{sl}_3(F)$ representation and $v \in V$ be a highest weight vector. Then, $\langle E_{2,1}, E_{3,1}, E_{3,2}\rangle.v$ is an irreducible subrepresentation of V.*

Proof. From above, it is clear that $W := \langle E_{2,1}, E_{3,1}, E_{3,2}\rangle.v$ is a subrepresentation. Let v have weight $\alpha \in \mathfrak{h}^*$. Then, it must be that W_α is one-dimensional since the action will reduce the weight relative to ℓ defined in the proof of 4.8. Assume W is not irreducible, so $W = W' \oplus W''$. However, the action of $\mathfrak{h}$ and projection onto either component commutes, so $W_\alpha = W'_\alpha \oplus W''_\alpha$. Thus, it must be that either W'_α or W''_α is zero and so v either lies in W' or in W'', and thus either $W = W'$ or $W = W''$. $\square$

4.12. **Corollary.** *Any irreducible representation of $\mathfrak{sl}_3(F)$ has a unique highest weight vector, up to scalar multiple.*

Now, similar to $\mathfrak{sl}_2(F)$, we notice that, given a highest weight vector, say $v \in \mathfrak{g}_{\epsilon_1-\epsilon_3}$, we have a collection of vectors $v, E_{2,1}.v, E_{2,1}^2.v, \ldots$ until $E_{2,1}^k.v = 0$ for some $k \in \mathbb{N}$. From this, we note that $\operatorname{span}\{E_{2,1}, E_{1,2}, [E_{1,2}, E_{2,1}]\}$ is isomorphic to $\mathfrak{sl}_2(F)$ via the isomorphism

$$e \mapsto E_{1,2}, f \mapsto E_{2,1}, h \mapsto [E_{1,2}, E_{2,1}]$$

This leads us to the proposition

4.13. **Proposition.** *The eigenvalues of $H_{1,2} := [E_{1,2}, E_{2,1}]$ on the subspace*

$$W = \bigoplus_k V_{\alpha+k(\epsilon_2-\epsilon_1)}$$

are integral and symmetric with respect to 0.

Proof. We note that W is a $\mathfrak{sl}_2(F)$-representation via the isomorphism given above. Thus, since all eigenvalues of $h \in \mathfrak{sl}_2(F)$ are integral and symmetric with respect to 0 for all $\mathfrak{sl}_2(F)$-representations, that must still be true in this case. $\square$

Of course, this process generalizes, and to each root we can associate a copy of $\mathfrak{sl}_2(F)$.

5. Beginning of a Structure Theory: Root Space Decomposition

We now seek to understand the structure of a general semisimple Lie algebra.

5.1. **Definition.** A subalgebra of a Lie algebra $\mathfrak{g}$ consisting of (nonzero) semisimple elements is called a *toral subalgebra*

5.2. **Lemma.** *A toral subalgebra of a Lie algebra* $\mathfrak{g}$ *is abelian.*

Proof. Let T be a toral subalgebra of $\mathfrak{g}$. We wish to show that $[x, T] = 0$ for all $x \in T$, which is the same as showing $\mathrm{ad}_x = 0$ (where ad is taken over T). Since x is semisimple, so is ad_x and since we are working over an algebraically closed field, ad_x is diagonalizable. Thus, we need only show that all the eigenvalues of ad_x are 0.

Assume that $[x, y] = ay$ for some $0 \neq a \in F$ and $y \in T$. Then, $\mathrm{ad}_y x = -ay$ is an eigenvector of ad_y with eigenvalue 0. We may also write x uniquely as a linear combination of eigenvectors of ad_y, say $\{-ay, v_1, \ldots, v_{n-1}\}$, since y is semisimple,

$$x = -c_0 ay + \sum_{i=1}^{n-1} c_i v_i \Longrightarrow -ay = \mathrm{ad}_y x = \sum_{i=1}^{n-1} c_i \lambda_i v_i$$

where λ_i is the eigenvalue of v_i. However, the set $\{-ay, v_1, \ldots, v_{n-1}\}$ is a basis, so it is linearly independent and $-ay$ cannot be written as a linear combination of $\{v_1, \ldots, v_{n-1}\}$. Thus, we have arrived at a contradiction. $\square$

5.3. **Definition.** A toral subalgebra $\mathfrak{h}$ not properly included in any other is called a *maximal toral subalgebra of* $\mathfrak{g}$.

5.4. **Example.** Consider $\mathfrak{g} = \mathfrak{sl}_2(F)$. Then, $\mathfrak{h} = \langle h \rangle$ is a maximal toral subalgebra.

5.5. **Remark.** If $\mathfrak{g}$ is a semisimple Lie algebra, it must have at least one element whose semisimple part is nonzero. Otherwise, $\mathfrak{g}$ would be nilpotent by Engel's theorem. Thus, $\mathfrak{g}$ contains nonzero toral subalgebras and thus contains nonzero maximal toral subalgebras.

5.6. **Definition.** Let $\mathfrak{h}$ be a maximal toral subalgebra for semisimple Lie algebra $\mathfrak{g}$. Then, for $\alpha \in \mathfrak{h}^*$, we define

$$\mathfrak{g}_\alpha := \{g \in \mathfrak{g} \mid [h, g] = \alpha(h)g, \forall h \in \mathfrak{h}\}$$

5.7. **Definition.** Let $\mathfrak{h}$ be a maximal toral subalgebra for semisimple Lie algebra $\mathfrak{g}$. Then, we define the *roots of* $\mathfrak{g}$ *relative to* $\mathfrak{h}$ to be

$$\Phi := \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}$$

5.8. **Theorem.** *A semisimple Lie algebra* $\mathfrak{g}$ *admits a decomposition*

$$\mathfrak{g} = C_{\mathfrak{g}}(\mathfrak{h}) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid [h, g] = \alpha(h) g, \forall h \in \mathfrak{h}\}$$

for any maximal toral subalgebra $\mathfrak{h}$ *of* $\mathfrak{g}$*.*

5.9. **Example.** Let $\mathfrak{g} = \mathfrak{sl}_2(F)$. The decomposition above corresponds to the decomposition of $\mathfrak{sl}_2(F)$ considered as an $\mathfrak{sl}_2(F)$-modules (the *regular representation*). Indeed, if $\mathfrak{h} = \mathrm{span}\{h\}$ for standard basis (e, f, h), then $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ and

$$\mathfrak{sl}_2(F) \cong \mathfrak{h} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_2$$

where $\mathfrak{g}_2 = \mathrm{span}\{e\}$ since $[h, e] = 2e$ and $\mathfrak{g}_{-2} = \mathrm{span}\{f\}$. In fact, a similar decomposition is easy to produce for $\mathfrak{g} = \mathfrak{sl}_n(F)$ with the standard basis construction.

5.10. **Remark.** In the example above, it is relatively straightforward to compute $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. In fact, this statement ends up being true for all semisimple Lie algebras. We wish to show this is true. Furthermore, we end up finding that the roots Φ completely characterize the structure of $\mathfrak{g}$. We will show that this is true in the remaining parts of this section.

5.11. **Proposition.** *The* ad_h*-eigenvector decomposition defines an* $\mathfrak{h}^*$*-grading on* $\mathfrak{g}$*.*

Proof. Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$ for $\alpha, \beta \in \mathfrak{h}^*$. Then, for $h \in \mathfrak{h}$,

$$\mathrm{ad}_h([x, y]) = [[h, x], y] + [x, [h, y]] = \alpha(h)[x, y] + \beta(h)[x, y] = (\alpha + \beta)(h)[x, y]$$

Thus, $[x, y] \in \mathfrak{g}_{\alpha+\beta}$. □

5.12. **Corollary.** *If* $x \in \mathfrak{g}_\alpha$ *for* $0 \neq \alpha \in \mathfrak{h}^*$*, then* ad_x *is nilpotent.*

Proof. For some $n \in \mathbb{N}$, $\mathfrak{g}_{n\alpha} = 0$, so $\mathrm{ad}_x^n = 0$ using the grading as above. □

5.13. **Proposition.** *Given* $\alpha, \beta \in \mathfrak{h}^*$ *with* $\alpha + \beta \neq 0$*, then, relative to the Killing form* κ *of* $\mathfrak{g}$*,* $\mathfrak{g}_\alpha$ *is orthogonal to* $\mathfrak{g}_\beta$*.*

Proof. Let $h \in \mathfrak{h}$ such that $(\alpha + \beta)(h) \neq 0$. Then, we get

$$\begin{aligned}
\alpha(h)\kappa(x, y) &= \kappa([h, x], y) \\
&= -\kappa([x, h], y) \\
&= -\kappa(x, [h, y]) \qquad \text{by the associativity of the Killing form.} \\
&= -\beta(h)\kappa(x, y)
\end{aligned}$$

Thus, $(\alpha + \beta)(h)\kappa(x, y) = 0$. (Recall that the Killing form is associative follows from the definition of the bracket on $\mathfrak{gl}(V)$.) □

5.14. **Corollary.** *The restriction of the Killing form to* $C_{\mathfrak{g}}(\mathfrak{h})$ *is nondegenerate.*

Proof. Since $\mathfrak{g}$ is semisimple, then the Killing form on $\mathfrak{g}$ is nondegenerate (see [Hum72, §5.1] or [See18, §6]). However, by the proposition above, $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0$ is orthogonal to all other $\mathfrak{g}_\alpha$'s with respect to the Killing form. Thus, it must be that, for $z \in \mathfrak{g}_0$,

$$\kappa(z, \mathfrak{g}_0) = 0 \Longrightarrow \kappa(z, \mathfrak{g}) = 0 \Longrightarrow z = 0$$

by the non-degeneracy of the Killing form on $\mathfrak{g}$. Thus, the Killing form is non-degenerate on $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h})$. □

5.15. **Proposition.** *Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. Then, $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h})$.*

Proof. We present only an outline of the proof given in [Hum72, §8.2].

(a) $C_{\mathfrak{g}}(\mathfrak{h})$ contains the semisimple and nilpotent parts of its elements since $x \in C_{\mathfrak{g}}(\mathfrak{h}) \Longrightarrow \mathrm{ad}_x(\mathfrak{h}) = 0$.
(b) All semisimple elements of $C_{\mathfrak{g}}(\mathfrak{h})$ lie in $\mathfrak{h}$ since $x \in C_{\mathfrak{g}}(\mathfrak{h}) \Longrightarrow \mathfrak{h} + Fx$ is toral and $\mathfrak{h}$ is maximal.
(c) The restriction of κ to $\mathfrak{h}$ is nondegenerate by the above items since $[x, \mathfrak{h}] = 0$ and ad_x is nilpotent $\Longrightarrow \kappa(x, \mathfrak{h}) = 0$.
(d) $C_{\mathfrak{g}}(\mathfrak{h})$ is nilpotent by Engel's theorem since all its elements are ad-nilpotent.
(e) $\mathfrak{h} \cap [C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})] = 0$ since $[\mathfrak{h}, C_{\mathfrak{g}}(\mathfrak{h})] = 0 \Longrightarrow \kappa(\mathfrak{h}, [C_g(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})]) = 0$.
(f) $C_{\mathfrak{g}}(\mathfrak{h})$ is abelian.
(g) $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$, otherwise $C_{\mathfrak{g}}(\mathfrak{h})$ contains a nonzero nilpotent element by above.

□

5.16. **Corollary.** *The Killing form restricted to $\mathfrak{h}$ is non-degenerate.*

Proof. Define map $\theta_h : \mathfrak{h} \to \mathfrak{h}$ via

$$\theta_h(k) = \kappa(h, k).$$

Since the Killing for is non-degenerate on $\mathfrak{h}$, the map $h \mapsto \theta_h$ is an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$ since the nondegeneracy gives trivial kernel and dimension count gives surjectivity. □

Thus, a maximal toral subalgebra (or a Cartan subalgebra) is *self-centralizing*. Now, we seek to understand the structure of the roots.

5.17. **Lemma.** *Given $\alpha \in \Phi$ of semisimple Lie algebra $\mathfrak{g}$ (over $\mathbb{C}$) and $0 \neq x \in \mathfrak{g}_\alpha$, then $-\alpha \in \Phi$ and there is a $y \in \mathfrak{g}_{-\alpha}$ such that $\{x, y, [x, y]\}$ spans a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.*

Proof. If $-\alpha \notin \Phi$, then, for all $\beta \in \Phi$,

$$\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \Longrightarrow \kappa(\mathfrak{g}_\alpha, \mathfrak{g}) = 0$$

by the orthogonality in 5.13, contradicting the non-degeneracy of κ.

Next, we note that $\alpha \neq 0$, so there is a $t \in \mathfrak{h}$ such that $\alpha(t) \neq 0$. Thus, for any $y \in \mathfrak{g}_{-\alpha}$,

$$\kappa(t, [x, y]) = \kappa([t, x], y) = \alpha(t)\kappa(x, y) \neq 0 \Longrightarrow [x, y] \neq 0$$

Now, we observe that $[x, y] \in \mathfrak{h} = \mathfrak{g}_0$ and x, y are simultaneous eigenvectors for $\mathrm{ad}[x, y]$, since they are for all elements of $\mathrm{ad}\,\mathfrak{h}$. Thus, $\mathrm{span}\{x, y, [x, y]\}$ is a Lie subalgebra of $\mathfrak{g}$.

Let $h := [x, y]$ and let S be our Lie subalgebra. To see that S is isomorphic to $\mathfrak{sl}_2$, we first observe that $\alpha(h) \neq 0$. Assume otherwise.

$$[h, x] = \alpha(h)x = 0 = -\alpha(h)y = [h, y]$$

thus showing that S is a solvable Lie subalgebra. Therefore, $[S, S]$ would be a nilpotent Lie subalgebra (see [Hum72, Cor 4.1C] or [See18]) and so, in particular, $\mathrm{ad}([x, y])$ is both semisimple and nilpotent, thus implying $\mathrm{ad}[x, y] = 0 \Longrightarrow [x, y] \in Z(S) = 0$, which is a contradiction. Thus, it must be that S is a 3-dimensional complex Lie algebra with $[S, S] = S$, which immediately implies that $S \cong \mathfrak{sl}_2(\mathbb{C})$. □

In fact, this will hold over more general fields, which we will show below, but this method provides quick insight into the guiding idea behind the representation theory of semisimple Lie algebras: understanding roots and the representation theory of $\mathfrak{sl}_2$. Luckily, we already understand the latter from the section above.

More generally, we have the following orthogonality proposition.

5.18. **Proposition.** [Hum72, Prop 8.3] *Let $\Phi \subseteq \mathfrak{h}^*$ be as above for semisimple Lie algebra $\mathfrak{g}$ over . Then,*

(a) Φ spans $\mathfrak{h}^*$.
(b) If $\alpha \in \Phi$, then $-\alpha \in \Phi$.
(c) If $\alpha \in \Phi, x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$, then $[x, y] = \kappa(x, y)t_\alpha$ where $t_\alpha \in \mathfrak{h}^*$ is the unique element such that $\alpha(h) = \kappa(t_\alpha, h)$ for all $h \in \mathfrak{h}$.
(d) Let $\alpha \in \Phi$. Then, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one dimensional, with basis t_α.
(e) $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$ for $\alpha \in \Phi$.
(f) If $\alpha \in \Phi$ and x_α is any nonzero element of $\mathfrak{g}_\alpha$, then there exists $y_\alpha \in L_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]$ span a three dimensional simple subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(F)$ via

$$x_\alpha \mapsto e, y_\alpha \mapsto f, h_\alpha \mapsto h$$

(g) $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$ and $h_\alpha = -h_{-\alpha}$.

5.19. **Remark.** This proposition will later show that Φ satisfies the axioms of a "root system" and is thus justifies the terminology "roots of $\mathfrak{g}$ relative to $\mathfrak{h}$."

5.20. **Example.** Let us demonstrate the result for $\mathfrak{g} = \mathfrak{sl}_2(F)$ since we have already shown what needs to be shown in the previous section. Then, $\mathfrak{h} = \operatorname{span}\{h\} \cong F$ and so $\mathfrak{h}^* = \operatorname{span}\{h^*\} \cong F$ where $h^*(h) = 1, h^*(e) = 0 = h^*(f)$ and $\Phi = \{\pm 2h^*\}$ since

$$\mathfrak{sl}_2 = \mathfrak{h} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2}$$

In this case, (a) and (b) above are immediate. For (c), we check

$$\kappa(e, f) = \operatorname{tr}(\operatorname{ad}_e \operatorname{ad}_f) = 4$$

and

$$[e, f] = h \Longrightarrow t_2 = \frac{1}{4}h^*$$

Indeed, one easily checks $\kappa(\frac{1}{4}h, \lambda h) = \lambda \cdot 2$ for all $\lambda \in F$.

For (d), $[\mathfrak{g}_2, \mathfrak{g}_{-2}] = \operatorname{span}\{[e, f]\} = \operatorname{span}\{h\}$ is one dimensional with basis given by $\frac{1}{4}h$.

For (e), $2 \cdot \frac{1}{4}h^*(h) = \frac{1}{16}\kappa(h, h) = \frac{1}{2}$.

For (f), the result is trivial. For (g), we note

$$\frac{2\frac{1}{4}h}{\frac{1}{16}\kappa(h, h)} = \frac{\frac{1}{2}h}{\frac{1}{2}} = h$$

Proof. For (a), assume Φ does not span $\mathfrak{h}^*$. Then, there is a $0 \neq h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in \Phi$. However, this gives, for all $\alpha \in \Phi$,

$$[h, \mathfrak{g}_\alpha] = 0 \Longrightarrow [h, \mathfrak{g}] = 0 \Longleftrightarrow h \in Z(\mathfrak{g}) = 0$$

which is a contradiction.

(b) is already proven in the lemma above.

For (c), take $\alpha \in \Phi, x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$. Then,

$$\kappa(h, [x, y]) = \kappa([h, x], y) = \alpha(h)\kappa(x, y) = \kappa(t_\alpha, h)\kappa(x, y) = \kappa(\kappa(x, y)t_\alpha, h) = \kappa(h, \kappa(x, y)t_\alpha)$$

and so, subtracting the two sides, we have $[x, y] - \kappa(x, y)t_\alpha$ is perpendicular to all $h \in \mathfrak{h}$. However, since κ is non-degenerate, this means that $[x, y] - \kappa(x, y)t_\alpha = 0$.

For (d), we see that t_α spans $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ assuming $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$ by part (c). Thus, we need only show that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$. Following the same technique as the proof of the lemma above, assume $\kappa(x, \mathfrak{g}_{-\alpha}) = 0$. Then, $\kappa(x, \mathfrak{g}) = 0$ is a contradiction since $\mathfrak{g}$ is semisimple. Thus, there is a $y \in \mathfrak{g}_{-\alpha}$ such that $\kappa(x, y) \neq 0 \Longrightarrow [x, y] \neq 0$.

For (e), assuming $\alpha(t_\alpha) = 0 \Longrightarrow [t_\alpha, x] = 0 = [t_\alpha, y]$ for all $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$. However, we can find an x, y such that $\kappa(x, y) \neq 0$ and, multiplying by scalars, we may assume $\kappa(x, y) = 1 \Longrightarrow [x, y] = t_\alpha$ by (c). Thus, using the same line of reasoning in the lemma, if we let $S = \operatorname{span}\{x, y, t_\alpha\}$, then $\operatorname{ad}_\mathfrak{g} t_\alpha = 0 \Longrightarrow t_\alpha \in Z(S) = 0$, which is a contradiction.

For (f), let $0 \neq x_\alpha \in \mathfrak{g}_\alpha$ and find a $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that

$$\kappa(x_\alpha, y_\alpha) = \frac{2}{\kappa(t_\alpha, t_\alpha)}$$

using (e) and the fact $\kappa(x_\alpha, \mathfrak{g}_{-\alpha}) \neq 0$. Then, by (c),

$$\begin{aligned} h_\alpha := \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} \Longrightarrow & [x_\alpha, y_\alpha] = h_\alpha \\ & [h_\alpha, x_\alpha] = \frac{2}{\alpha(t_\alpha)}[t_\alpha, x_\alpha] = \frac{2\alpha(t_\alpha)}{\alpha(t_\alpha)} x_\alpha = 2x_\alpha \\ & [h_\alpha, y_\alpha] = -\frac{2}{\alpha(t_\alpha)}[t_\alpha, y_\alpha] = -\frac{2\alpha(t_\alpha)}{\alpha(t_\alpha)} y_\alpha = -2y_\alpha \end{aligned}$$

Thus, $\{x_\alpha, y_\alpha, h_\alpha\}$ span a three dimensional subalgebra of $\mathfrak{g}$ with the $\mathfrak{sl}_2(F)$ relations.

For (g), we note that $t_\alpha = -t_{-\alpha}$ since $\kappa(-t_{-\alpha}, h) = \alpha(h) = \kappa(t_\alpha, h)$ for all $h \in \mathfrak{h}$. Thus, since $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$,

$$-h_{-\alpha} = -\frac{2t_{-\alpha}}{\kappa(t_{-\alpha}, t_{-\alpha})} = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} = h_\alpha$$

□

As noted above, the correspondence of $\mathfrak{sl}_2(F)$ inside any semisimple Lie algebra to a root is incredibly important, so much so that we will define a new notation to refer to such a copy of $\mathfrak{sl}_2(F)$.

5.21. **Definition.** For a pair of roots $\alpha, -\alpha$, let $\mathfrak{sl}(\alpha) \cong \mathfrak{sl}_2(F)$ be a subalgebra of the form presented in the lemma or part (f) of the proposition above, that is, a subalgebra generated by $\{x_\alpha, y_\alpha, [x_\alpha, y_\alpha]\}$ for $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$.

With this terminology, we have the following proposition

5.22. **Proposition.** *Given a semisimple Lie algebra $\mathfrak{g}$ with root α, $\mathfrak{g}$ may be considered as an $\mathfrak{sl}(\alpha)$-module via the adjoint action, that is, for $a \in \mathfrak{sl}(\alpha), g \in \mathfrak{g}$*

$$a.g = \operatorname{ad}_a g = [a, g]$$

Proof. This module is simply a restriction of the adjoint representation. □

5.23. **Definition.** Let $\mathfrak{g}$ be a semisimple Lie algebra with roots Φ. Let $\beta \in \Phi$ or $\beta = 0$. Then, subspace

$$\bigoplus_{\substack{k \in F \\ \beta + k\alpha \in \Phi}} \mathfrak{g}_{\beta + k\alpha}$$

is an *α-root string through β*.

Note that such a root string is an $\mathfrak{sl}(\alpha)$-module. Now, using this notion, we prove the following.

5.24. **Lemma.** *Let $\Phi \subseteq \mathfrak{h}^*$ be a set of roots for a semisimple Lie algebra $\mathfrak{g}$. Then, for $\alpha \in \Phi$, the only multiples of α which lie in Φ are $\pm\alpha$. Furthermore, $\mathfrak{g}_\alpha$ is one-dimensional for every root $\alpha \in \Phi$.*

Proof. Let

$$M := \mathfrak{h} \oplus \bigoplus_{k\alpha \in \Phi} \mathfrak{g}_{k\alpha}$$

be the α-root string through 0. If $c\alpha$ is a root, then $h_\alpha.x_{c\alpha} = 2c\alpha(h_\alpha)x_{c\alpha} = 2cx_{c\alpha}$, so h_α has $2c$ as an eigenvalue. However, the eigenvalues of h_α must be integral, so $2c \in \mathbb{Z}$. Now, by direct computation, $\mathfrak{sl}(\alpha)$ acts trivially on $\ker\alpha \subseteq \mathfrak{h}$. Next, given $\mathfrak{h} + \mathfrak{sl}(\alpha)$, we know that, as $\mathfrak{sl}(\alpha)$-modules,

$$\mathfrak{h} + \mathfrak{sl}(\alpha) \cong \ker\alpha \oplus \mathfrak{sl}(\alpha)$$

since $\dim\ker\alpha = \dim\mathfrak{h} - \dim\operatorname{im}\alpha = \dim\mathfrak{h} - 1$. Thus, by Weyl's theorem and the above computation,

$$\mathfrak{h} + \mathfrak{sl}(\alpha) \subseteq \mathfrak{h} \oplus \bigoplus_{k\alpha \in \Phi} \mathfrak{g}_{k\alpha} \cong \ker\alpha \oplus \mathfrak{sl}(\alpha) \oplus N$$

where N is some complementary submodule. Now, let V be an irreducible representation of $\mathfrak{sl}_2(F)$ with dimension k. Then, if k is even, there is a $v \in V$ with $h_\alpha.v = 0 \implies \alpha(v) = 0 \implies v \in \ker\alpha$ by virtue of the classification of $\mathfrak{sl}_2(F)$ representations. However, $\ker\alpha \cap V = 0 \implies v = 0$, which is a contradiction. Thus, the only even weights occuring in M are 0 and ± 2, which tells us that 2α cannot be a root. However, this also means that $\frac{1}{2}\alpha$ is not a root, so 1 also cannot be a weight of M. Thus, since the number of irreducible $\mathfrak{sl}_2(F)$-module summands is precisely $\dim M_0 + \dim M_1$, it must be that N contains no irreducible $\mathfrak{sl}_2(F)$-submodules, and is thus trivial. Therefore,

$$M \cong \ker\alpha \oplus \mathfrak{sl}(\alpha)$$

but $\ker\alpha \cong \mathfrak{g}_0$ can only be a direct sum of trivial $\mathfrak{sl}(\alpha)$-modules by the classification of $\mathfrak{sl}_2(F)$ representations. Thus, $\mathfrak{g}_\alpha$ is one-dimensional since $\mathfrak{sl}(\alpha) = \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. ∎

5.25. **Proposition.** [Hum72, Prop 8.4] *Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$. Then,*

(a) $\beta(h_\alpha) \in \mathbb{Z}$
(b) There exist integers $r, q \geq 0$ such that $\beta + k\alpha \in \Phi$ if and only if $k \in \mathbb{Z}$ and $-r \leq k \leq q$. Furthermore, $\beta(h_\alpha) = r - q$.

(c) $\beta - \beta(h_\alpha)\alpha \in \Phi$
(d) If $\alpha + \beta \in \Phi$*, then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$*.*
(e) $\mathfrak{g}$ *is generated (as a Lie algebra) by the root spaces* $\mathfrak{g}_\alpha$*.*

Proof. We will now look at the action of $\mathfrak{sl}(\alpha)$ on

$$M := \bigoplus_{\substack{k\in\mathbb{Z}\\ \beta+k\alpha\in\Phi}} \mathfrak{g}_{\beta+k\alpha}.$$

Since $h_\alpha.w = \beta(h_\alpha)w$ for $w \in \mathfrak{g}_\beta$, it must be that $\beta(h_\alpha) \in \mathbb{Z}$. From the lemma above, there can be no k such that $\beta + k\alpha = 0$ since, otherwise, β would be a multiple of α. Thus, the action of h_α on each one-dimensional root space is given by

$$h_\alpha.v = (\beta(h_\alpha) + 2k)v$$

for $v \in \mathfrak{g}_{\beta+k\alpha}$. Because the weights differ by even integers, it cannot be that both 0 and 1 occur as a weight of this form, and either can only appear once. Thus, M must be an irreducible $\mathfrak{sl}(\alpha)$-module and the weights must be symmetric around 0. Let the heighest weight be given by $\beta + q\alpha$ and the lowest weight by $\beta - r\alpha$. Thus,

$$-(\beta(h_\alpha) + 2q) = \beta(h_\alpha) - 2r \Longrightarrow \beta(h_\alpha) = r - q$$

which also tells us that

$$\beta - \beta(h_\alpha)\alpha = \beta - (r-q)\alpha \in \Phi$$

since $-r \le q - r \le q$.

Now, for (d), let $v \in \mathfrak{g}_\beta$ and $\mathrm{ad}_{x_\alpha} x_\beta = 0$. Then, x_β is a highest weight vector of M with weight $\beta(h_\alpha)$. However, we also know that if $\alpha + \beta$ is a root, then h_α acts on $\mathfrak{g}_{\alpha+\beta}$ by eigenvalue $(\alpha+\beta)h_\alpha = \beta(h_\alpha) + 2$, which tells us that $\beta(h_\alpha)$ is not the highest weight, a contradiction. Thus, it must be that $\mathrm{ad}_{x_\alpha} x_\beta \neq 0 \Longrightarrow [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Finally, by virtue of (d), $\mathfrak{g}$ is generated as a Lie algebra by the root spaces since, given generator $t_\alpha \in \mathfrak{g}_\alpha$ and $t_\beta \in \mathfrak{g}_\beta$, if $\alpha+\beta \in \Phi$, $[t_\alpha, t_\beta]$ is a non-zero scalar multiple of $t_{\alpha+\beta}$. $\square$

5.26. **Definition.** The integers $\beta(h_\alpha)$ are called the *Cartan integers.*

At this point, we now understand a lot of the structure constants of $\mathfrak{g}$ using the basis given by the root space decomposition. Namely, the action of $\mathfrak{h}$ is determined by the roots, $[x_\alpha, x_\beta]$ is determined by the roots for $\beta \neq \pm\alpha$, and $[x_\alpha, x_{-\alpha}]$ is a scalar multiple of h_α. Finally, we wish to prove Φ has a well-understood strucutre.

5.27. **Lemma.** *For* $\alpha, \beta \in \Phi$*, we have*

$$\kappa(t_\alpha, t_\beta) = \sum_{\gamma\in\Phi} \gamma(t_\alpha)\gamma(t_\beta)$$

Proof. By definition,
$$\kappa(t_\alpha, t_\beta) = \operatorname{tr}(\operatorname{ad}_{t_\alpha} \operatorname{ad}_{t_\beta})$$
However, as a map, $\operatorname{ad}_{t_\alpha} \operatorname{ad}_{t_\beta} \colon \mathfrak{g} \to \mathfrak{g}$ has $\operatorname{ad}_{t_\alpha} \operatorname{ad}_{t_\beta}(h) = 0$ for all $h \in \mathfrak{h}$ and, for $x \in \mathfrak{g}_\gamma$, we have
$$x \mapsto [t_\alpha, [t_\beta, x]] = [t_\alpha, \gamma(t_\beta)x] = \gamma(t_\alpha)\gamma(t_\beta)x$$
So, the trace of the map will be the sum in the proposition. □

5.28. **Lemma.** *If $\alpha, \beta \in \Phi$, then $\kappa(h_\alpha, h_\beta) \in \mathbb{Z}$ and $\kappa(t_\alpha, t_\beta) \in \mathbb{Q}$.*

Proof. We first realize that
$$\kappa(h_\alpha, h_\beta) = \operatorname{tr}(\operatorname{ad}_{h_\alpha} \operatorname{ad}_{h_\beta}) = \sum_{\gamma \in \Phi} \gamma(h_\alpha)\gamma(h_\beta)$$
since the trace is the sum of the eigenvalues with multiplicity as in 5.27. However, all these eigenvalues are integers, so $\kappa(h_\alpha, h_\beta) \in \mathbb{Z}$. Next, we compute
$$\kappa(t_\alpha, t_\beta) = \kappa\left(\frac{\kappa(t_\alpha, t_\alpha)}{2} h_\alpha, \frac{\kappa(t_\beta, t_\beta)}{2} h_\beta\right) = \frac{\kappa(t_\alpha, t_\alpha)\kappa(t_\beta, t_\beta)}{4} \kappa(h_\alpha, h_\beta)$$
However,
$$\frac{1}{\kappa(t_\alpha, t_\alpha)} = \frac{1}{4}\kappa\left(\frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}, \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}\right) = \frac{1}{4}\kappa(h_\alpha, h_\alpha) \in \mathbb{Q}$$
and thus $\kappa(t_\alpha, t_\beta) \in \mathbb{Q}$ since it is a product of rational numbers. □

5.29. **Definition.** Let us define form $(\cdot, \cdot) \colon \mathfrak{h}^* \times \mathfrak{h}^* \to F$ via
$$(\gamma, \delta) := \kappa(t_\gamma, t_\delta)$$
for $\gamma, \delta \in \mathfrak{h}^*$

5.30. **Proposition.** *Such a form is non-degenerate and bilinear.*

Proof. Since the Killing form is non-degenerate on $\mathfrak{g}$, it is non-degenerate on $\mathfrak{h}$ and thus $(\cdot, \cdot)$ is non-degenerate on $\mathfrak{h}^*$. Furthermore, $(\cdot, \cdot)$ inherits its bilinearity from the Killing form. □

5.31. **Lemma.** *If $\beta \in \Phi$ and $\{\alpha_1, \ldots, \alpha_n\}$ is a basis of roots for $\mathfrak{h}^*$, then β is a rational linear combination of the α_i's.*

Proof. Consider $\beta = \sum c_i \alpha_i$. Then, we compute
$$(\beta, \alpha_j) = \sum_i (\alpha_i, \alpha_j) c_i$$
which gives us n equations with n unknowns (the c_i's) with rational coefficients.
$$(\beta, \alpha_1) = (\alpha_1, \alpha_1)c_1 + (\alpha_2, \alpha_1)c_2 + \cdots + (\alpha_n, \alpha_1)c_n$$
$$(\beta, \alpha_2) = (\alpha_1, \alpha_2)c_1 + (\alpha_2, \alpha_2)c_2 + \cdots + (\alpha_n, \alpha_2)c_n$$

$$\vdots$$
$$(\beta, \alpha_n) = (\alpha_1, \alpha_n)c_1 + (\alpha_2, \alpha_n)c_2 + \cdots + (\alpha_n, \alpha_n)c_n$$

Since $(\cdot, \cdot)$ is non-degenerate, the associated matrix is non-degenerate and thus invertible. Furthermore, since all the coefficients are rational, its solutions will be rational. Thus, $c_i \in \mathbb{Q}$. $\square$

5.32. **Definition.** Let $\mathfrak{g}$ be a semisimple Lie algebra with roots Φ. Then, we define $E_{\mathbb{Q}} \subseteq \mathfrak{h}^*$ to be the $\mathbb{Q}$-subspace spanned by all the roots Φ.

By virtue of the proposition above, we know that all roots are $\mathbb{Q}$-linear combinations of other roots, so $E_{\mathbb{Q}}$ is independent of choice of roots for a basis. We now wish to show $(\cdot, \cdot)$ has more structure on $E_{\mathbb{Q}}$.

5.33. **Proposition.** *The restriction of $(\cdot, \cdot)$ to $E_{\mathbb{Q}}$ is rational and positive-definite*

Proof. From lemma 5.28, we know that $\kappa(t_\alpha, t_\beta) \in \mathbb{Q}$ for any roots $\alpha, \beta \in \Phi$, but Φ $\mathbb{Q}$-spans $E_{\mathbb{Q}}$ by the proposition above, and so we can pick a basis of roots for $E_{\mathbb{Q}}$, say $\{\alpha_1, \ldots, \alpha_n\}$. Thus, for $\gamma, \delta \in E_{\mathbb{Q}}$,

$$(\gamma, \delta) = \left(\sum c_i\alpha_i, \sum d_j\alpha_j\right) = \sum_{i,j} c_i d_j \cdot (\alpha_i, \alpha_j) = \sum_{i,j} c_i d_j \kappa(t_{\alpha_i}, t_{\alpha_j})$$

but $c_i, d_j \in \mathbb{Q}$ by definition of $E_{\mathbb{Q}}$. Finally, given $\lambda \in E_{\mathbb{Q}}$,

$$(\lambda, \lambda) = \kappa(t_\lambda, t_\lambda) \overset{5.27}{=} \sum_{\gamma \in \Phi} \gamma(t_\lambda)^2 = \sum_{\gamma \in \Phi} (\gamma, \lambda)^2 \geq 0$$

If $(\lambda, \lambda) = 0$, then $\gamma(t_\lambda) = 0$ for all roots γ, and so $\lambda = 0$ by 5.18 (e). Thus, $(\cdot, \cdot)$ is positive-definite. $\square$

5.34. **Lemma.** $\beta(h_\alpha) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$

Proof. Using the fact that $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$, we see

$$\frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \frac{2\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = \kappa\left(t_\beta, \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}\right) = \kappa(t_\beta, h_\alpha) = \beta(h_\alpha)$$

$\square$

5.35. **Definition.** Let $E := \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}}$.

5.36. **Remark.** Using standard linear algebra, the form $(\cdot, \cdot)|_{E_{\mathbb{Q}}}$ extends canonically to E and is still positive-definite. Thus, E is a Euclidean space.

Thus, we see that we have already shown Φ has a incredibly useful structure:

5.37. **Proposition.** *Given a semisimple Lie algebra $\mathfrak{g}$, Φ is a* root system *in E, that is*

(a) Φ spans E and $0 \notin \Phi$.

(b) If $\alpha \in \Phi$, then the only multiples of α in Φ are $\pm\alpha$.
(c) If $\alpha, \beta \in \Phi$, then $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$.
(d) If $\alpha, \beta \in \Phi$, then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Proof. (a) is 5.18 (a) and the definition of Φ.
(b) is 5.24.
(c) is a rephrasing of 5.25 (c) in light of lemma 5.34.
(d) is a rephrasing of 5.25 (a) in light of lemma 5.34. □

Thus, the structure of Φ is given by the geometry of root systems, which are incredibly symmetric structures that are well-understood. There are many introductions to root systems, including [Hum72, Ch III], [Car05, Ch 5], and [See17]. From this point onwards, a working knowledge of root systems is assumed. We see that the structure of *irreducible* root systems corresponds to that of simple Lie algebras:

5.38. **Proposition.** *Let $\mathfrak{g}$ be a simple Lie algebra, with maximal toral subalgebra $\mathfrak{h}$ and $\Phi \subseteq \mathfrak{h}^*$. Then, Φ is an irreducible root system.*

Proof. Assume the root system is reducible into orthogonal root systems: $\Phi = \Phi_1 \cup \Phi_2$. Then, take $\alpha \in \Phi_1, \beta \in \Phi_2$. We see that

$$(\alpha + \beta, \alpha) \neq 0 \neq (\alpha + \beta, \beta) \implies \alpha + \beta \notin \Phi$$

since $\alpha + \beta$ does not lie strictly in Φ_1 or Φ_2. Therefore, it must be that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$$

by the root space decomposition. Observe

(a) For all $\beta \in \Phi_2$, $\mathfrak{g}_\beta$ centralizes the subalgebra generated by $\mathfrak{g}_\alpha$, say K, which must be proper since $Z(\mathfrak{g}) = 0$ by the simplicity of $\mathfrak{g}$.
(b) For all $\alpha \in \Phi_1$, $\mathfrak{g}_\alpha$ normalizes K since K is generated by $\mathfrak{g}_\alpha$

Therefore, K is normalized by all $\mathfrak{g}_\gamma$ for $\gamma \in \Phi$ and thus by all of $\mathfrak{g}$ because $\mathfrak{g}$ is generated by its root spaces (5.25). Thus, K is a proper ideal of $\mathfrak{g}$, violating the simplicity of $\mathfrak{g}$. □

5.39. **Corollary.** [Hum72, p 74] *Let $\mathfrak{g}$ be a semisimple Lie algebra with maximal toral subalgebra $\mathfrak{h}$ and root system Φ. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_t$ is the decomposition of $\mathfrak{g}$ into simple ideals, then $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$ is a maximal toral subalgebra of $\mathfrak{g}_i$ and the corresponding (irreducible) root system Φ_i may be regarded canonically as a subsystem of Φ in such a way that $\Phi = \Phi_1 \cup \cdots \cup \Phi_t$ is the decomposition of Φ into its irreducible components.*

Proof. Since $\mathfrak{g}$ is semisimple, we know that it has a decomposition into simple ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_t$ and if $\mathfrak{h}$ is a maximal toral subalgebra of $\mathfrak{g}$, then $\mathfrak{h} = \mathfrak{g}_1 \cap \mathfrak{h} \oplus \cdots \oplus \mathfrak{g}_t \cap \mathfrak{h}$ since, $x = x_1 + \cdots + x_t \in \mathfrak{h}$ is semisimple and each $x_i \in \mathfrak{g}_i$ is semisimple by 2.18, so $x_i \in \mathfrak{g}_i \cap \mathfrak{h}$.

Now, let $\mathfrak{h}_i := \mathfrak{g}_i \cap \mathfrak{h}$. Then, $\mathfrak{h}_i$ is a maximal toral subalgebra in $\mathfrak{g}_i$ since any toral subalgebra of $\mathfrak{g}_i$ larger than $\mathfrak{h}_i$, say $\mathfrak{h}_i'$ would be toral in $\mathfrak{g}$ and centralize all $\mathfrak{h}_j, j \neq i$, and so $\mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_i' \oplus \cdots \oplus \mathfrak{h}_t$ would be a toral subalgebra of $\mathfrak{g}$ larger than $\mathfrak{h}$.

Let Φ_i be the root system of $\mathfrak{g}_i$ relative to $\mathfrak{h}_i$. If $\alpha \in \Phi_i$, we can view α as a linear function on $\mathfrak{h}$ by defining $\alpha(\mathfrak{h}_j) = 0$ for $j \neq i$. So, α is thus a root of $\mathfrak{g}$ relative to $\mathfrak{h}$ with $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_i$.

Conversely, let $\alpha \in \Phi$. Then, $[\mathfrak{h}_i, \mathfrak{g}_\alpha] \neq 0$ for some i and so $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_i$, otherwise $\mathfrak{h}$ would centralize $\mathfrak{g}_\alpha$. From this, we get that $\alpha|_{\mathfrak{h}_i}$ is a root of $\mathfrak{g}_i$ relative to $\mathfrak{h}_i$. Note that α cannot be a root of another system since $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_i$ and $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ for all $j \neq i$.

Thus, we can decompose $\Phi = \Phi_1 \cup \cdots \cup \Phi_t$ into irreducible components. $\square$

From the above, by understanding all the irreducible root systems, a purely geometric task, we gain understanding of all the possible root systems a simple Lie algebra might have. Furthermore, one can show, with more work, that any irreducible root system corresponds to a simple Lie algebra. This is a consequence of results such as Serre's Theorem [Hum72, §18.3] or by the existence of the Chevalley basis [Hum72, §25].

6. A Return to Representations of $\mathfrak{sl}_3(F)$

As seen above, we have successfully generalized our results about $\mathfrak{sl}_3(F)$ and a little bit more. In particular, we see that the subspace of all diagonal matrices generalizes to a maximal toral subalgebra $\mathfrak{h}$ and the ad_h-action always yields integral eigenvalues on the root spaces.

6.1. **Example.** The roots $\Phi = \{\epsilon_i - \epsilon_j\}$ of $\mathfrak{sl}_3(F)$ form a root system as described in the proposition concluding the previous chapter. To see this explicitly, one checks by straightforward computation that parts (a) and (b) of 5.37 are true. Then, we note that $\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\}$ forms a basis for Φ. We then check

$$\begin{cases} [E_{i,j}, E_{j,i}] = E_{i,i} - E_{j,j} \\ \kappa(E_{i,j}, E_{j,i}) = 6 \end{cases} \implies t_{\epsilon_i - \epsilon_j} = \frac{1}{6}(E_{i,i} - E_{j,j})$$

and so

$$\begin{aligned}(\epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_2) &= \frac{1}{36}\kappa(E_{1,1} - E_{2,2}, E_{1,1} - E_{2,2}) \\ &= \frac{1}{36}(\epsilon_1 - \epsilon_2)(E_{1,1} - E_{2,2}) \\ &= \frac{1}{36}(1+1)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{18} \\
(\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3) &= \frac{1}{36}\kappa(E_{1,1} - E_{2,2}, E_{2,2} - E_{3,3}) \\
&= \frac{1}{36}(\epsilon_1 - \epsilon_2)(E_{2,2} - E_{3,3}) \\
&= \frac{1}{36}(0 - 1) \\
&= -\frac{1}{36} \\
(\epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_3) &= \frac{1}{36}(\epsilon_2 - \epsilon_3)(E_{2,2} - E_{3,3}) \\
&= \frac{1}{36}(1 + 1) \\
&= \frac{1}{18}
\end{aligned}
$$

and so

$$(\epsilon_1 - \epsilon_2) - \frac{2(\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3)}{(\epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_3)}(\epsilon_2 - \epsilon_3) = \epsilon_1 - \epsilon_2 - (-1)(\epsilon_2 - \epsilon_3) = \epsilon_1 - \epsilon_3$$

which shows (c) and demonstates (d). Indeed, for (d), one can see that $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = -1$ or 2 for α, β in the basis.

(Note, for the general $\mathfrak{sl}_n(F)$ case, one can prove that $\kappa(E_{i,j}, E_{j,i}) = 2n$ and thus $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = 0, \pm 1$ for all $\beta \neq \pm\alpha$.)

In fact, the root system above can be visualized as the configuration A_2 and, with that insight, we can visualize the weight spaces of any $\mathfrak{sl}_3(F)$ representation as a lattice in a hexagonal configuration. Thus, we can visualize a root string as a line through the lattice. For instance, for $\alpha \in \Phi$, Add images

$$W := \bigoplus_{\substack{k \in \mathbb{Z} \\ \alpha + k(\epsilon_2 - \epsilon_1)}} \mathfrak{g}_{\alpha + k(\epsilon_2 - \epsilon_1)}$$

is symmetric about the line $\{L \mid \langle E_{1,1} - E_{2,2}, L\rangle = 0\}$ and is thus preserved by reflection across said line.

6.2. **Proposition.** [FH91, Prop 12.15] *All eigenvalues of any irreducible finite-dimensional representation of $\mathfrak{sl}_3(F)$ must lie in the weight lattice $\Lambda_W \subseteq \mathfrak{h}^*$, that is, the lattice generated by the ϵ_i and must be congruent modulo the lattice $\Lambda_R \subseteq \mathfrak{h}^*$ generated by the $\epsilon_i - \epsilon_j$.*

6.3. **Proposition.** [FH91, Prop 12.18] *Let V be any irreducible, finite-dimensional representation of $\mathfrak{sl}_3(F)$. Then, for some $\alpha \in \Lambda_W \subseteq \mathfrak{h}^*$, the set of eigenvalues occurring in V is exactly the set of linear functionals*

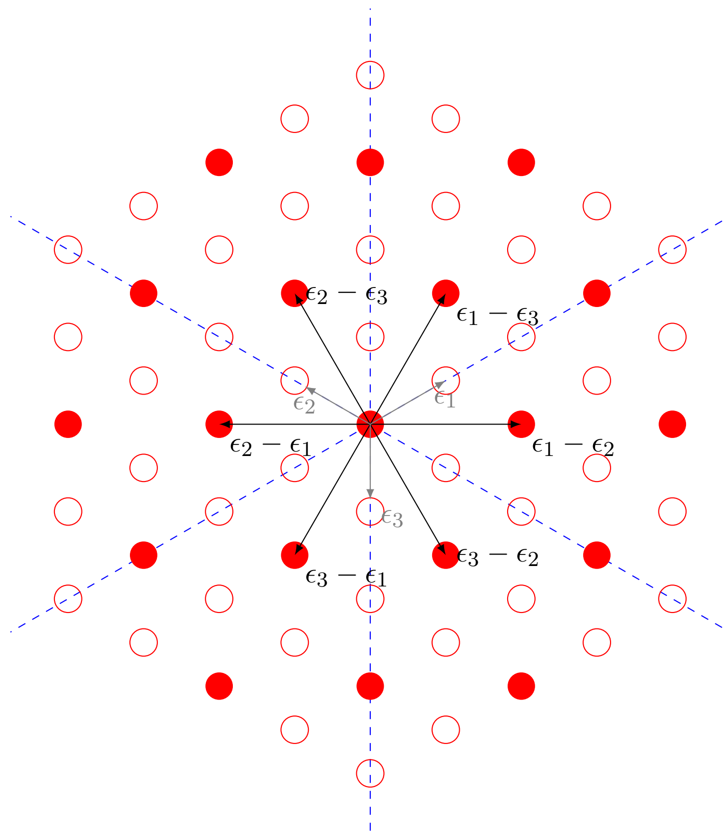

FIGURE 1. Weight lattice for $\mathfrak{sl}_3$. Choice of simple roots α_1, α_2 are given by black arrows.

congruent to α modulo the lattice Λ_R and lying in the hexagon with vertices the images of α under the group generated by reflections in the lines $\{L \mid \langle E_{i,i} - E_{j,j}, L\rangle = 0\}$.

Fill in proofs for these propositions.

From this, we see that, given our choice of ℓ, it must be that any highest weight vector will lie in the region cut out by the inequalities $\langle E_{1,1} - E_{2,2}, L\rangle \geq 0$ and $\langle E_{2,2} - E_{3,3}, L\rangle \geq 0$. In other words, it must be of the form

$$(a+b)\epsilon_1 + b\epsilon_2 = a\epsilon_1 - b\epsilon_3$$

Thus, we have reason to believe.

6.4. **Theorem.** *For any pair of natural numbers a, b, there exists a unique irreducible, finite-dimensional representation $\Gamma_{a,b}$of $\mathfrak{sl}_3(F)$ with highest weight $a\epsilon_1 - b\epsilon_3$.*

To prove this theorem, we will construct such representations explicitly. Thus, let us look at some examples.

6.5. **Example.** Consider the standard representation of $\mathfrak{sl}_3(\mathbb{C})$ on $V \cong \mathbb{C}^3$ with basis $\{e_1, e_2, e_3\}$ with associated eigenvalues ϵ_1, ϵ_2, and ϵ_3 since, for

$$h = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} e_i = a_i e_i = \epsilon_i(h) e_i$$

Consider also its dual, V^*, with dual basis $\{e_1^*, e_2^*, e_3^*\}$ and associated eigen-

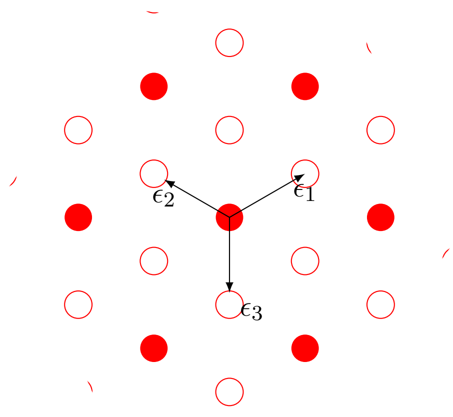

FIGURE 2. Weight diagram for V, the standard representation of $\mathfrak{sl}_3(\mathbb{C})$

values $-\epsilon_1, -\epsilon_2$, and $-\epsilon_3$.

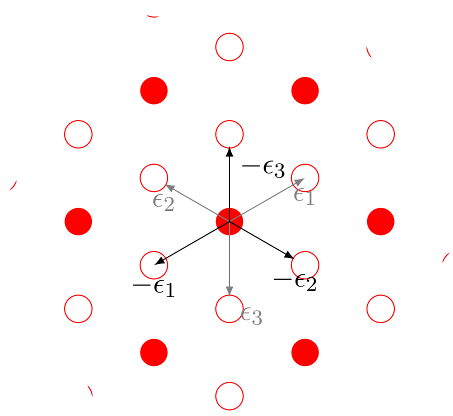

FIGURE 3. Weight diagram for V^*, the dual representation of the standard representation of $\mathfrak{sl}_3(\mathbb{C})$

Using weight diagrams , one can see that $\mathrm{Sym}^2 V$ and $\mathrm{Sym}^2 V^*$, whose weights are the pairwise sums of the weights of V and V^*, respectively, are both irreducible. A more interesting analysis comes from examining $V \otimes V^*$.

Actually insert these

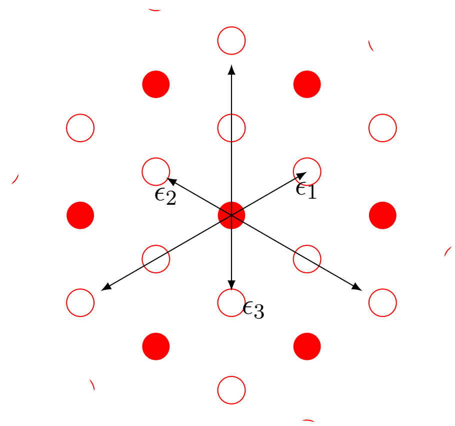

FIGURE 4. Weight diagram for $\mathrm{Sym}^2 V^*$

One can quickly check that the weights of $V \otimes V^*$ are $\{\epsilon_1 \pm \epsilon_2, \epsilon_1 \pm \epsilon_3, \epsilon_2 \pm \epsilon_3\}$, each with multiplicity 1, and 0 with multiplicity 3. Since the linear map

$$v \otimes u^* \mapsto \langle v, u^* \rangle = u^*(v)$$

has a non-trivial kernel. One can easily check that this map is equivalent to the trace since $V \otimes V^* \cong \mathrm{Hom}(V, V) \cong M_3(\mathbb{C})$ and thus the kernel is all traceless matrices, ie the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$, which is an irreducible $\mathfrak{sl}_3(\mathbb{C})$-representation. Thus,

$$V \otimes V^* \cong \mathfrak{sl}_3(\mathbb{C}) \oplus \mathbb{C}$$

We will also need the following lemma

6.6. **Lemma.** *Given representations V and W with highest weight vectors v, w with weights α, β, respectively, then $v \otimes w \in V \otimes W$ is a highest weight vector of weight $\alpha + \beta$.*

Proof. Consider that, for $E_{i,j}$ with $i \leq j$, we get

$$E_{i,j}.(v \otimes w) = E_{i,j}.v \otimes w + v \otimes E_{i,j}.w = 0 + 0 = 0$$

and, for $H \in \mathfrak{h}$, we get

$$\begin{aligned} H.(v \otimes w) &= H.v \otimes w + v \otimes H.w \\ &= \alpha(H) v \otimes w + v \otimes \beta(H) w \\ &= (\alpha(H) + \beta(H))(v \otimes w) \end{aligned}$$

□

Thus, from the example and the lemma, we get

6.7. **Proposition.** *Given standard representation V, we have the following*

(a) V has a highest weight vector with weight ϵ_1.
(b) V^* has a highest weight vector with weight $-\epsilon_3$.

(c) $\mathrm{Sym}^n V$ *has a highest weight vector with weight* $n \cdot \epsilon_1$.
(d) $\mathrm{Sym}^n V^*$ *has a highest weight vector with weight* $-n \cdot \epsilon_3$.

Proof of Theorem 6.4. Let V be the standard representation of $\mathfrak{sl}_3(F)$. Then, $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ contains an irreducible subrepresentation generated by the highest weight vector with weight $a\epsilon_1 - b\epsilon_3$.

To show uniqueness, let V, W be two $\mathfrak{sl}_3(F)$-representations with highest weight α and corresponding highest weight vectors v, w. Then. consider $(v, w) \in V \oplus W$ will be a highest weight vector with highest weight α. Let U be the irreducible subrepresentation generated by (v, w). Then, we have non-zero projections between irreducibles,

$$\pi_1 \colon U \to V, \quad \pi_2 \colon U \to W$$

which, by Schur's lemma, must be isomorphisms. Thus, $V \cong U \cong W$. $\square$

In fact, we can more explicitly construct the representations $\Gamma_{a,b}$, most easily using the Weyl Character Formula, which has not been proven yet. However, the final result will be given by first considering the contraction map

$$\begin{aligned} \iota_{a,b} \colon \mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^* &\to \mathrm{Sym}^{a-1} V \otimes \mathrm{Sym}^{b-1} V^* \\ (v_1 \cdots v_a) \otimes (v_1^* \cdots v_b^*) &\mapsto \sum \langle v_i, v_j^* \rangle (v_1 \cdots \hat{v_i} \cdots v_a) \otimes (v_1^* \cdots \hat{v_j^*} \cdots v_b^*) \end{aligned}$$

Such a map is surjective and the codomain cannot have eigenvalue $a\epsilon_1 - b\epsilon_3$, so $\Gamma_{a,b} \subseteq \ker \iota_{a,b}$. In fact, using the Weyl Character Formula, we immediately conclude

6.8. **Proposition.** $\ker \iota_{a,b} = \Gamma_{a,b}$.

Also, as an immediate corollary, we conclude

6.9. **Corollary.** *If* $b \leq a$, $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^* \cong \bigoplus_{i=0}^{b} \Gamma_{a-i,b-i}$ *and similarly if* $a \leq b$.

7. More Structure Theory of Semisimple Lie Algebras

We state some results here without proof since they can be superseded by Serre's theorem. However, we wish to establish that maximal toral subalgebras and Cartan subalgebras coincide for semisimple Lie algebras. Along the way, we establish a few other distinguished subalgebras of semisimple Lie algebras which appear throughout the literature on Lie algebras.

7.1. **Proposition.** [Hum72, p 74] *Let* $\mathfrak{g}$ *be a semisimple Lie algebra,* $\mathfrak{h} \subseteq \mathfrak{g}$ *a maximal toral subalgebra, and* Φ *the root system of* $\mathfrak{g}$ *relative to* $\mathfrak{h}$*. Fix a base* Δ *of* Φ*. Then,* $\mathfrak{g}$ *is generated as a Lie algebra by the root spaces* $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ *for* $\alpha \in \Delta$*. Equivalently,* $\mathfrak{g}$ *is generated by arbitrary nonzero root vectors* $x_\alpha \in \mathfrak{g}_\alpha, y_\alpha \in \mathfrak{g}_{-\alpha}$ *for* $\alpha \in \Delta$*.*

7.2. **Theorem.** [Hum72, p 75] *Let $\mathfrak{g}, \mathfrak{g}'$ be simple Lie algebra over a field F with respective maximal toral subalgebras $\mathfrak{h}, \mathfrak{h}'$ and corresponding root systems Φ, Φ'. Suppose there is an isomorphism of Φ onto Φ' given by $\alpha \mapsto \alpha'$ inducing $\pi \colon \mathfrak{h} \to \mathfrak{h}'$. Fix a base $\Delta \subseteq \Phi$ so that $\Phi' = \{\alpha' \mid \alpha \in \Delta\}$ is a base of Φ'. For each $\alpha \in \Delta, \alpha' \in \Delta'$, choose arbitrary (nonzero) $x_\alpha \in \mathfrak{g}_\alpha, x'_\alpha \in \mathfrak{g}'_\alpha$, that is, choose an arbitrary Lie algebra isomoprhism $\pi_\alpha \colon \mathfrak{g}_\alpha \to \mathfrak{g}'_\alpha$. Then, there eixsts a unique isomorphism $\pi \colon \mathfrak{g} \to \mathfrak{g}'$ extending $\pi \colon \mathfrak{h} \to \mathfrak{h}'$ and extending all of the π_α, $\alpha \in \Delta$.*

$$\begin{tikzcd}
\mathfrak{h} \arrow[r] \arrow[d, "\pi"] & \mathfrak{g} \arrow[d, dashed, "\pi"] & \mathfrak{g}_\alpha \arrow[l] \arrow[d, "\pi_\alpha"] \\
\mathfrak{h}' \arrow[r] & \mathfrak{g}' & \mathfrak{g}'_\alpha \arrow[l]
\end{tikzcd}$$

7.3. **Proposition.** [Hum72, p 77] *Given $\mathfrak{g}$ as above, but not necessarily simple, fix $0 \neq x_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in \Delta$ and let $y_\alpha \in \mathfrak{g}_{-\alpha}$ satisfy $[x_\alpha, y_\alpha] = h_\alpha$. Then, there exists an automorphism σ of $\mathfrak{g}$ of order 2, satisfying, for $\alpha \in \Delta, h \in \mathfrak{h}$, $\sigma(x_\alpha) = -y_\alpha, \sigma(y_\alpha) = -x_\alpha, \sigma(h) = -h$.*

Proof. This follows from the fact that any automorphism of Φ can be extended to $\operatorname{Aut}\Phi$, so in particular, the map sending each root to its negative is in $\operatorname{Aut}\Phi$. ☐

7.4. **Definition.** Let $t \in \operatorname{End} V$, a be a root of the characteristic polynomial of t with multiplicity m, and $V_a := \ker(t - a \cdot 1)^m$. Then, given an $x \in \mathfrak{g}$, a Lie algebra over algebraically closed field F , we call $\mathfrak{g}_0(\operatorname{ad} x)$ for $\operatorname{ad} x \in \operatorname{End}\mathfrak{g}$ an *Engel subalgebra*.

7.5. **Lemma.** [Hum72, p 78] *If $a, b \in F$, then $[\mathfrak{g}_a(\operatorname{ad} x), \mathfrak{g}_b(\operatorname{ad} x)] \subseteq \mathfrak{g}_{a+b}(\operatorname{ad} x)$. In particular, $\mathfrak{g}_0(\operatorname{ad} x)$ is a subalgebra of $\mathfrak{g}$, and when* $\operatorname{char} F = 0, a \neq 0$*, each element of $\mathfrak{g}_a(\operatorname{ad} x)$ is* ad*-nilpotent. Thus, an Engel subalgebra is actually an algebra.*

Proof. Compute the expansion

$$(\operatorname{ad} x - a - b)^m [y, z] = \sum_{i=0}^{m} \binom{m}{i} [(\operatorname{ad} x - a)^i (y), (\operatorname{ad} x - b)^{m-i}(z)]$$

for $y \in \mathfrak{g}_a(\operatorname{ad} x), z \in \mathfrak{g}_b(\operatorname{ad} x)$. For sufficiently large m the expression will be 0. ☐

7.6. **Lemma.** [Hum72, p 79] *Let $\mathfrak{g}'$ be a subalgebra of $\mathfrak{g}$. Choose $z \in \mathfrak{g}'$ such that $\mathfrak{g}_0(\operatorname{ad} z)$ is minimal in the collection of all $\mathfrak{g}_0(\operatorname{ad} x)$ for $x \in \mathfrak{g}'$. Suppose that $\mathfrak{g}' \subseteq \mathfrak{g}_0(\operatorname{ad} z)$. Then, $\mathfrak{g}_0(\operatorname{ad} z) \subseteq \mathfrak{g}_0(\operatorname{ad} x)$ for all $x \in \mathfrak{g}'$.*

7.7. **Lemma.** [Hum72, p 79] *If $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$ containing an Engel subalgebra, then $N_{\mathfrak{g}}(\mathfrak{g}') = \mathfrak{g}'$. In particular, Engel subalgebras are self-normalizing.*

7.8. **Definition.** A *Cartan subalgebra* (CSA) of a Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra which equals its normalizer in $\mathfrak{g}$.

7.9. **Theorem.** [Hum72, p 80] *Let $\mathfrak{h}$ be a subalgebra of Lie algebra $\mathfrak{g}$. Then, $\mathfrak{h}$ is a CSA of $\mathfrak{g}$ if and only if $\mathfrak{h}$ is a minimal Engel subalgebra.*

7.10. **Corollary.** [Hum72, p 80] *Let $\mathfrak{g}$ be semisimple over F with* $\operatorname{char} F = 0$. *Then, the CSA's of $\mathfrak{g}$ are precisely the maximal toral subalgebras of $\mathfrak{g}$.*

7.11. **Proposition.** [Hum72, p 81] *Let $\phi\colon \mathfrak{h} \to \mathfrak{h}'$ be a surjective Lie algebra homomorphism.*

(a) If $\mathfrak{h}$ is a CSA of $\mathfrak{g}$, then $\phi(\mathfrak{h})$ is a CSA of $\mathfrak{g}'$.
(b) If $\mathfrak{h}'$ is a CSA of $\mathfrak{g}'$ and $K = \phi^{-1}(\mathfrak{h}')$, then any CSA $\mathfrak{h}$ of K is also a CSA of $\mathfrak{g}$.

7.12. **Definition.** A *Borel subalgebra* of a Lie algebra $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{g}$.

7.13. **Definition.** Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{C}$ and let $\mathfrak{h}$ be a CSA of $\mathfrak{g}$. Then, $\mathfrak{g}$ has *triangular decomposition*

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$$

where $\mathfrak{n}^- = \bigoplus_{\alpha\in\Phi^-} \mathfrak{g}_\alpha$ and $\mathfrak{n} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$.

7.14. **Proposition.** *The $\mathfrak{n}^-$ and $\mathfrak{n}$ in the triangular decomposition are subalgebras of $\mathfrak{g}$.*

Proof. The sum of two positive roots is still positive. So, for $\alpha, \beta \in \Phi^+$, we have

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \subseteq \mathfrak{n}$$

and so $\mathfrak{n}$ is a subalgebra. Similarly for $\mathfrak{n}^-$. □

7.15. **Proposition.** *(a) $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is a Borel subalgebra, called* standard *relative to $\mathfrak{h}$.*
(b) $\mathfrak{n}$ is an ideal of $\mathfrak{b}$.
(c) $\mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$.

Proof. For (a), we first observe that

$$[\mathfrak{b}, \mathfrak{b}] = [\mathfrak{h} + \mathfrak{n}, \mathfrak{h} + \mathfrak{n}] \subseteq \mathfrak{h} + \mathfrak{n} = \mathfrak{b}$$

because $\mathfrak{h}, \mathfrak{n}$ are subalgebras and $[\mathfrak{h}, \mathfrak{n}] \subseteq \mathfrak{n}$. Thus, $\mathfrak{b}$ is a subalgebra.

Let $\mathfrak{g}'$ be any subalgebra of $\mathfrak{g}$ properly containing $\mathfrak{b}$. Since it is a subalgebra containing $\mathfrak{h}$, $\mathfrak{g}'$ must be stable under the $\operatorname{ad}\mathfrak{h}$ action, and so it must include some $\mathfrak{g}_\alpha$ with $\alpha \prec 0$. However, then $\mathfrak{g}'$ contains a copy of $\mathfrak{sl}_2$ generated by $\{x_\alpha, y_\alpha, [x_\alpha, y_\alpha]\}$ for $x_\alpha \in \mathfrak{g}_{-\alpha}, y_\alpha \in \mathfrak{g}_\alpha$. Since $\mathfrak{sl}_2$ is simple, $\mathfrak{g}'$ cannot be solvable.

For (b), we observe

$$[\mathfrak{n}, \mathfrak{b}] = [\mathfrak{n}, \mathfrak{n} + \mathfrak{h}] \subseteq \mathfrak{n}$$

For (c), we compute

$$\mathfrak{b}/\mathfrak{n} = (\mathfrak{h} + \mathfrak{n})/\mathfrak{n} \cong \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{n}) \cong \mathfrak{h}$$

because $\mathfrak{h} \cap \mathfrak{n} = 0$. □

7.16. **Proposition.** [Hum72, pp 83–84]

(a) *If* $\mathfrak{b}$ *is a Borel subalgebra of* $\mathfrak{g}$*, then* $\mathfrak{b} = N_{\mathfrak{g}}(\mathfrak{b})$.
(b) *If* $\operatorname{rad} \mathfrak{g} \neq \mathfrak{g}$*, then the Borel subalgebras of* $\mathfrak{g}$ *are in natural one-to-one correspondence with those of the semisimple Lie algebra* $\mathfrak{g}/\operatorname{rad} \mathfrak{g}$.
(c) *Let* $\mathfrak{g}$ *be semisimple with CSA* $\mathfrak{h}$ *and root system* Φ*. All standard Borel subalgebras of* $\mathfrak{g}$ *relative to* $\mathfrak{h}$ *are conugate under inner automorphisms of* $\operatorname{Aut} \mathfrak{g}$.

7.17. **Theorem.** [Hum72, p 84] *The Borel subalgebras of an arbitrary Lie algebra* $\mathfrak{g}$ *are all conjugate under inner automorphisms of* $\operatorname{Aut} \mathfrak{g}$.

7.18. **Corollary.** [Hum72, p 84] *The Cartan subalgebras of an arbitrary Lie algebra* $\mathfrak{g}$ *are conjugate under inner automorphisms of* $\operatorname{Aut} \mathfrak{g}$.

8. Universal Enveloping Algebra

In this section, we will construct an associative algebra $\mathcal{U}(\mathfrak{g})$ such that the representation theory of $\mathcal{U}(\mathfrak{g})$ is the same as the representation theory of $\mathfrak{g}$.

8.1. **Definition.** Let V be a finite dimensional vector space over a field F. Then, we define

$$\begin{aligned} T^0V &= F \\ T^1V &= V \\ T^2V &= V \otimes V \\ &\vdots \\ T^nV &= \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}} \end{aligned}$$

and say $T(V) = \bigsqcup T^kV$ equipped with the product

$$T^mV \times T^nV \to T^{m+n}V$$
$$(v_1 \otimes \cdots \otimes v_m) \cdot (w_1 \otimes \cdots \otimes w_n) \mapsto v_1 \otimes \cdots \otimes v_m \otimes w_1 \otimes \cdots \otimes w_n$$

is called the *tensor algebra* on V.

8.2. **Proposition.** *Let V be a finite dimensional vector space and A an associative unital algebra over F. Then, we have the following universal property:*

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "\phi"'] & T(V) \arrow[d, dashed, "\exists!\Phi"] \\ & A \end{tikzcd}$$

In words, for any F-linear map $\phi\colon V \to A$ and $\iota\colon V \to T(V)$ sending $v \mapsto v \in T^1V$, there is a unique homomorphism of F-algebra $\Phi\colon T(V) \to A$ such that $\Phi \circ \iota = \phi$.

8.3. **Definition.** Let $\mathfrak{g}$ be a Lie algebra. Then, consider the ideal $J = \langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}\rangle$ in $T(\mathfrak{g})$. We define

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/J$$

and call such a quotient the *universal enveloping algebra* of $\mathfrak{g}$.

8.4. **Example.** Let $\mathfrak{g}$ be an n-dimensional abelian Lie algebra over $\mathbb{C}$ with basis $\{x_1, \dots, x_n\}$ and $[x_i, x_j] = 0$. Then, $J \trianglelefteq T(\mathfrak{g})$ is given by $J = \langle x \otimes y - y \otimes x \mid x, y \in \mathfrak{g}\rangle$. Therefore, $\mathcal{U}(\mathfrak{g})$ is commutative and generated by $1, x_1, \dots, x_n$. Thus, $\mathcal{U}(\mathfrak{g}) \cong \mathbb{C}[x_1, \dots, x_n]$.

8.5. **Proposition.** *Let $\mathfrak{g}$ be an arbitrary Lie algebra. Then, if $\iota\colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is a linear map satisyfing*

$$\iota([x, y]) = \iota(x)\iota(y) - \iota(y)\iota(x), \;\; x, y \in \mathfrak{g}$$

and $\phi\colon \mathfrak{g} \to A$ satisfies the same property for A an associative unital F-algebra, then we have the following universal property

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "\phi"'] & \mathcal{U}(\mathfrak{g}) \arrow[d, dashed, "\exists!\Phi"] \\ & A \end{tikzcd}$$

that is, there is a unique F-algebra homomorphism $\Phi\colon \mathcal{U}(\mathfrak{g}) \to A$ such that $\Phi \circ \iota = \phi$.

8.6. **Remark.** Note that we have not shown that ι is injective, but we will do so later with the PBW Theorem.

8.7. **Proposition.** *There is a bijective correspondence between representations $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ and representations $\phi\colon \mathcal{U}(\mathfrak{g}) \to \operatorname{End} V$. Corresponding representations are related by the condition*

$$\phi(\iota(x)) = \theta(x) \textit{ for all } x \in \mathfrak{g}$$

where $\iota\colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is the standard inclusion.

Proof. Given θ a representation of $\mathfrak{g}$, we have

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "\theta"'] & \mathcal{U}(\mathfrak{g}) \arrow[d, dashed, "\exists ! \phi"] \\ & \mathfrak{gl}(V) = \operatorname{End} V \end{tikzcd}$$

where $\phi \colon \mathcal{U}(\mathfrak{g}) \to \operatorname{End} V$ is a representation of $\mathcal{U}(\mathfrak{g})$.

Conversely, given ϕ a representation of $\mathcal{U}(\mathfrak{g})$, we know that the linear map $\iota \colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ has

$$\iota(x)\iota(y) - \iota(y)\iota(x) = \iota([x, y]) \Longrightarrow [\iota(x), \iota(y)] = \iota([x, y])$$

if we define $[\cdot, \cdot] \colon \mathcal{U}(\mathfrak{g}) \times \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ by $[x, y] = xy - yx$. Thus, $\iota \colon \mathfrak{g} \to [\mathcal{U}(\mathfrak{g})]$, that is, $\mathcal{U}(\mathfrak{g})$ with the extra Lie algebra structure, is a Lie algebra homomorphism.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] & \mathcal{U}(\mathfrak{g}) \arrow[d, "\phi"] \\ & \operatorname{End} V \end{tikzcd} \Longrightarrow \begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, dashed, "\phi \circ \iota"'] & {[\mathcal{U}(\mathfrak{g})]} \arrow[d, "\phi"] \\ & \mathfrak{gl}(V) \end{tikzcd}$$

Then, $\phi \circ \iota$ is a representation $\square$

8.8. **Remark.** Note, the representations under consideration need not be finite-dimensional. In fact, it will turn out that a certain important class of representations called "Verma modules" will always be infinite-dimensional.

This construction will lead us to the all-important PBW Theorem, which tells us how to obtain a basis for the universal enveloping algebra of a Lie algebra.

8.9. **Theorem** (Poincaré-Birkhoff-Witt Theorem)**.** [Car05, Theorem 9.4] *Let $\mathfrak{g}$ be a Lie algebra with basis $\{x_i \mid i \in I\}$. Let $<$ be a total order on the index set I. Let $\iota \colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ be the natural linear map from $\mathfrak{g}$ to its enveloping algebra. We define $y_i := \iota(x_i)$. Then, the elements*

$$y_{i_1}^{r_1} \cdots y_{i_n}^{r_n}$$

for all $n \geq 0$, all $r_i \geq 0$, and all $i_1, \ldots, i_n \in I$ with $i_1 < I_2 < \cdots < i_n$ form a basis for $\mathcal{U}(\mathfrak{g})$.

For the proof of this theorem, see [Car05, pp 155–159] or [Hum72, pp 93–94] (Note, [Hum72] gives an equivalent statement of the PBW Theorem and states this version as Corollary C). The PBW Theorem has many important corollaries such as

8.10. **Corollary.** *The map $\iota \colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is injective.*

Proof. By the PBW Theorem, the elements y_i for $i \in I$ are linearly independent. Since the $x_i \in \mathfrak{g}$ form a basis for $\mathfrak{g}$ and $\iota(x_i) = y_i$, it must be that $\ker \iota = \{0\}$. $\square$

8.11. **Corollary.** *The subspace $\iota(\mathfrak{g}) \subseteq [\mathcal{U}(\mathfrak{g})]$ is isomorphic to $\mathfrak{g}$ as a Lie subalgebra.*

Proof. From the corollary above, $\iota\colon \mathfrak{g} \to \iota(\mathfrak{g})$ is bijective and $y_i, i \in I$ form a basis of $\iota(\mathfrak{g})$. Furthermore,

$$[y_i, y_j] = [\iota(x_i), \iota(x_j)] = \iota([x_i, x_j]) \in \iota(\mathfrak{g})$$

Thus, $\iota(\mathfrak{g})$ is a Lie subalgebra of $[\mathcal{U}(\mathfrak{g})]$. $\square$

8.12. **Corollary.** *$\mathcal{U}(\mathfrak{g})$ has no zero-divisors.*

Proof. Take $0 \neq a, b \in \mathcal{U}(\mathfrak{g})$. Then,

$$\begin{aligned}
a &= \sum \lambda_{i_1,\dots,i_n,r_1,\dots,r_n} y_{i_1}^{r_1} \cdots y_{i_n}^{r_n} \\
&= \sum_{\substack{r_1+\cdots+r_n=r \\ \text{maximal degree}}} \lambda_{i_1,\dots,i_n,r_1,\dots,r_n} y_{i_1}^{r_1} \cdots y_{i_n}^{r_n} + \sum_{r_1+\cdots+r_n<r} \lambda_{i_1,\dots,i_n,r_1,\dots,r_n} y_{i_1}^{r_1} \cdots y_{i_n}^{r_n} \\
&= f(y_i) + \text{ sum of terms of smaller degree} \\
b &= \sum \mu_{i_1,\dots,i_n,r_1,\dots,r_n} y_{i_1}^{r_1} \cdots y_{i_n}^{r_n} \\
&= \sum_{\substack{r_1+\cdots+r_n=r \\ \text{maximal degree}}} \mu_{i_1,\dots,i_n,r_1,\dots,r_n} y_{i_1}^{r_1} \cdots y_{i_n}^{r_n} + \sum_{r_1+\cdots+r_n<r} \mu_{i_1,\dots,i_n,r_1,\dots,r_n} y_{i_1}^{r_1} \cdots y_{i_n}^{r_n} \\
&= g(y_i) + \text{ sum of terms of smaller degree}
\end{aligned}$$

Recall also, since $\iota(\mathfrak{g})$ is a Lie subalgebra,

$$y_i y_j - y_j y_i = [y_i, y_j] = \sum_k \nu_k y_k$$

and so why?

$$f(y_i) g(y_i) = (fg)(y_i) + \text{a sum of terms of smaller degree.}$$

Therefore,

$$ab = (fg)(y_i) + \text{a sum of terms of smaller degree.}$$

and since $f \neq 0 \neq g$, we get $fg \neq 0$. Thus, the PBW Theorem tells us that $ab \neq 0$. $\square$

9. Irreducible Modules of Semisimple Lie Algebras

9.1. **Lemma.** [Car05, Lemma 10.1] *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$, and let $\mathfrak{g}'$ be a subalgebra of $\mathfrak{g}$. Then, there exists a unique algebra* *homomorphism* $\theta\colon \mathcal{U}(\mathfrak{g}') \to \mathcal{U}(\mathfrak{g})$ *such that the following diagram commutes*

$$\begin{tikzcd}
\mathfrak{g}' \arrow[r, "\iota_{\mathfrak{g}'}"] \arrow[d, "i"] & \mathcal{U}(\mathfrak{g}') \arrow[d, "\theta"] \\
\mathfrak{g} \arrow[r, "\iota_{\mathfrak{g}}"] & \mathcal{U}(\mathfrak{g})
\end{tikzcd}$$

where i *is the embedding of* $\mathfrak{g}'$ *into* $\mathfrak{g}$ *and* $\iota_{\mathfrak{g}'}, \iota_{\mathfrak{g}}$ *are the standard embeddings of a Lie algebra into its universal enveloping algebra. Furthermore,* θ *is injective.*

Proof. For $x \in \mathfrak{g}'$, the definition

$$\theta(\iota_{\mathfrak{g}'}(x)) = \iota_{\mathfrak{g}}(i(x))$$

is well-defined and, since $\mathcal{U}(\mathfrak{g}')$ is generated by $\iota(\mathfrak{g}')(\mathfrak{g}')$, such a θ is uniquely determined. To see existence, define one notices

$$\begin{tikzcd}
\mathfrak{g}' \arrow[r] \arrow[d, "i"] & T(\mathfrak{g}') \arrow[r] \arrow[d, dashed, "i^*"] & T(\mathfrak{g}')/J' \cong \mathcal{U}(\mathfrak{g}') \arrow[d, dotted] \\
\mathfrak{g} \arrow[r] & T(\mathfrak{g}) \arrow[r] & T(\mathfrak{g})/J \cong \mathcal{U}(\mathfrak{g})
\end{tikzcd}$$

since $i^*(J') \subseteq J$, where J, J' are the ideals used to define the universal enveloping algebra.

To see θ is injective, let $x_1, \ldots, x_r$ be a basis for $\mathfrak{g}'$ and $0 \neq u \in \mathcal{U}(\mathfrak{g}')$. Then, u is a non-zero linear combination of monomials $x_1^{e_1} \cdots x_r^{e_r}$. Then, let $x_1, \ldots, x_r$ be part of a basis of $\mathfrak{g}$. The PBW theorem tells us that such a linear combination of monomials cannot be zero in $\mathcal{U}(\mathfrak{g})$. Thus, $\theta(u) \neq 0$ and so $\ker\theta = 0$. $\square$

9.2. **Definition.** Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ and let $\lambda \in \mathfrak{h}^*$. We define, for $\mathfrak{h} = \{h_1, \ldots, h_k\}$,

$$K_\lambda := \sum_{\alpha\in\Phi^+} \mathcal{U}(\mathfrak{g})x_\alpha + \sum_{i=1}^{k} \mathcal{U}(\mathfrak{g})(h_i - \lambda(h_i))$$

for basis $\{x_\alpha \mid \alpha \in \Phi\} \cup \{h_i \mid i = 1, \ldots, k\}$ of $\mathfrak{g}$. Since K_λ is a left ideal of $\mathcal{U}(\mathfrak{g})$, we also define

$$M(\lambda) := \mathcal{U}(\mathfrak{g})/K_\lambda$$

and call $M(\lambda)$ the *Verma module* determined by λ. Since x_α for $\alpha \in \Phi^+$ and $h_i - \lambda(h_i)$ for $i = 1, \ldots, k$ all lie in $\mathcal{U}(\mathfrak{b})$ for $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, we may also define

$$K'_\lambda := \sum_{\alpha\in\Phi^+} \mathcal{U}(\mathfrak{b})x_\alpha + \sum_{i=1}^{k} \mathcal{U}(\mathfrak{b})(h_i - \lambda(h_i))$$

which is a left ideal of $\mathcal{U}(\mathfrak{b})$.

9.3. **Proposition.** [Car05, Prop 10.4] *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with positive roots $\Phi^+ = \{\beta_1, \dots, \beta_N\}$ and basis $h_1, \dots, h_k, x_{\beta_1}, \dots, x_{\beta_N}$.*

(a) $\dim \mathcal{U}(\mathfrak{b})/K'_\lambda = 1$

(b) The elements

$$(h_1 - \lambda(h_1))^{s_1} \cdots (h_k - \lambda(h_k))^{s_k} x_{\beta_1}^{t_1} \dots x_{\beta_N}^{t_n}$$

with $s_i \geq 0, t_i \geq 0$, excluding the element with $s_i = t_i = 0$ for all i, form a basis for K'_λ.

Proof. We first wish to show the elements listed above form a basis for $\mathcal{U}(\mathfrak{b})$. To see this, one can define a partial ordering on the basis elements by saying

$$h_1^{s'_1} \cdots h_k^{s'_k} x_{\beta_1}^{t'_1} \cdots x_{\beta_N}^{t'_N} \prec h_1^{s_1} \cdots h_k^{s_k} x_{\beta_1}^{t_1} \cdots x_{\beta_N}^{t_N} \text{ if } s'_1 \leq s_1, \dots, s'_k \leq s_k, t'_1 = t_1, \dots, t'_N = t_N$$

Thus,

$$(h_1 - \lambda(h_1))^{s_1} \cdots (h_k - \lambda(h_k))^{s_k} x_{\beta_1}^{t_1} \dots x_{\beta_N}^{t_n} = h_1^{s_1} \cdots h_k^{s_k} x_{\beta_1}^{t_1} \cdots x_{\beta_N}^{t_N} + \text{strictly lower terms}$$

Thus, by induction on the exponents s_i, t_i and from the triangularity property above, elements of the form

$$(h_1 - \lambda(h_1))^{s_1} \cdots (h_k - \lambda(h_k))^{s_k} x_{\beta_1}^{t_1} \dots x_{\beta_N}^{t_n}, \quad s_i \geq 0, t_i \geq 0$$

span $\mathcal{U}(\mathfrak{b})$ and they are linearly independent.

By definition, it follows immedaitely that all of these elements except for the element with $s_i = 0, t_i = 0$ for all i are in K'_λ, but it is not clear if this exceptional element, which is equal to the unit element 1, is in K'_λ or not. We will show that 1 is not in K'_λ.

Consider the representation $\lambda \colon \mathfrak{h} \to \mathbb{C}$. Since $\mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$, one can lift to a 1-dimensional representation of $\mathfrak{b}$ with kernel $\mathfrak{n}$ agreeing with λ on $\mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$. From this, we get a 1-dimensional representation ρ of $\mathcal{U}(\mathfrak{b})$ which does

$$\begin{aligned} x_\alpha &\mapsto 0 & \alpha &\in \Phi^+ \\ h_i &\mapsto \lambda(h_i) & i &= 1, \dots, k \\ 1 &\mapsto 1 \end{aligned}$$

This gives us

$$\begin{cases} x_\alpha \in \ker \rho & \alpha \in \Phi^+ \\ h_i - \lambda(h_i) \in \ker \rho & i = 1, \dots, k \\ 1 \notin \ker \rho \end{cases} \implies \begin{cases} K'_\lambda \subseteq \ker \rho \\ 1 \notin \ker \rho \end{cases} \implies 1 \notin K'_\lambda$$

Therefore, both (a) and (b) follow. $\square$

9.4. **Proposition.** [Car05, Prop 10.5] $K_\lambda \cap \mathcal{U}(\mathfrak{n}^-) = 0$.

Proof.
$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b} \Longrightarrow \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-)\mathcal{U}(\mathfrak{b})$$
by the PBW theorem. Then,
$$\begin{aligned}
K_\lambda &= \sum_{\alpha\in\Phi^+} \mathcal{U}(\mathfrak{g})x_\alpha + \sum_{i=1}^{k} \mathcal{U}(\mathfrak{g})(h_i - \lambda(h_i)) \\
&= \sum_{\alpha\in\Phi^+} \mathcal{U}(\mathfrak{n}^-)\mathcal{U}(\mathfrak{b})x_\alpha + \sum_{i=1}^{k} \mathcal{U}(\mathfrak{n}^-)\mathcal{U}(\mathfrak{b})(h_i - \lambda(h_i)) \\
&= \mathcal{U}(n^-)\left(\sum_{\alpha\in\Phi^+} \mathcal{U}(\mathfrak{b})x_\alpha + \sum_{i=1}^{k} \mathcal{U}(\mathfrak{b})(h_i - \lambda(h_i))\right) \\
&= \mathcal{U}(\mathfrak{n}^-)K'_\lambda
\end{aligned}$$
Therefore, every element of K_λ is a linear combination of terms of the form
$$\underbrace{y_{\beta_1}^{r_1}\cdots y_{\beta_N}^{r_N}}_{\in\mathcal{U}(\mathfrak{n}^-)}\underbrace{(h_1-\lambda(h_1))^{s_1}\cdots(h_k-\lambda(h_k))^{s_k}x_{\beta_1}^{t_1}\cdots x_{\beta_N}^{t_N}}_{\in K'_\lambda}$$
where $y_\alpha = x_{-\alpha}$ for $\alpha \in \Phi^+$ and $r_i, s_i, t_i \geq 0$ with $(s_1, \ldots, s_k, t_1, \ldots, t_N) \neq (0, 0, \ldots, 0)$. However, the PBW Theorem also tells us that no non-zero element of $\mathcal{U}(\mathfrak{n}^-)$ can be a a linear combination of such terms, since its basis must be of the form $f_{\beta_1}^{r_1}\cdots f_{\beta_N}^{r_N}$. Therefore, $K_\lambda \cap \mathcal{U}(\mathfrak{n}^-) = 0$. ☐

9.5. **Theorem.** [Car05, Theorem 10.6]*Let $m_\lambda := 1+K_\lambda \in M(\lambda) \cong \mathcal{U}(\mathfrak{g})/K_\lambda$. Then,*

- *(a) Each element of $M(\lambda)$ is uniquely expressible in the form um_λ for some $u \in \mathcal{U}(\mathfrak{n}^-)$, that is, $M(\lambda)$ is cyclically generated by m_λ as a $\mathcal{U}(\mathfrak{n}^-)$-module.*
- *(b) Moreover, the elements $y_{\beta_1}^{r_1}\cdots y_{\beta_N}^{r_n}m_\lambda$ for all $r_i \geq 0$ form a basis for $M(\lambda)$.*

Proof. Since $1 \mapsto m_\lambda$ under the quotient map $\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})/K_\lambda \cong M(\lambda)$, every element of $M(\lambda)$ has the form um_λ for some $u \in \mathcal{U}(\mathfrak{g})$.

Now, we have basis for $\mathfrak{g}$,
$$\{x_\alpha, y_\alpha \mid \alpha \in \Phi^+\} \cup \{h_i \mid i = 1, \ldots, k\}$$
thus, by the PBW theorem, giving us a basis for $\mathcal{U}(\mathfrak{g})$ of elements of the form
$$y_{\beta_1}^{r_1}\cdots y_{\beta_N}^{r_N}h_1^{s_1}\cdots h_k^{s_k}x_{\beta_1}^{t_1}\cdots x_{\beta_N}^{t_N} \quad r_1, s_i, t_i \geq 0$$
Now, observe
$$\begin{aligned}
x_{\beta_i}m_\lambda &= x_{\beta_i}(1+K_\lambda) = x_{\beta_i} + K_\lambda = 0 + K_\lambda = 0 \in M(\lambda) \quad \text{since } x_{\beta_i} \in K_\lambda \\
h_i m_\lambda &= \lambda(h_i)m_\lambda
\end{aligned}$$

and so

$$y_{\beta_1}^{r_1}\cdots y_{\beta_N}^{r_N}h_1^{s_1}\cdots h_k^{s_k}x_{\beta_1}^{t_1}\cdots x_{\beta_N}^{t_N}m_\lambda = \begin{cases} 0 & \text{if any } t_i > 0 \\ y_{\beta_1}^{r_1}\cdots y_{\beta_N}^{r_N}\gamma m_\lambda & \text{for some } \gamma\in\mathbb{C} \text{ if all } t_i = 0 \end{cases}$$

Therefore, since um_λ is a linear combination of elements of the form of the product above, such a linear combination simplifies to a linear combination of elements of the form

$$y_{\beta_1}^{r_1}\cdots r_{\beta_N}^{r_N}m_\lambda,\ r_i\geq 0$$

and so elements of this form span $M(\lambda)$.

To see linear independence, let

$$\sum_{r_1,\ldots,r_N}\gamma_{r_1,\ldots,r_N}y_{\beta_1}^{r_1}\cdots y_{\beta_N}^{r_N}m_\lambda = 0,\ \gamma_{r_1,\ldots,r_N}\in\mathbb{C}$$

and so, by the proposition above,

$$\sum_{r_1,\ldots,r_N}\gamma_{r_1,\ldots,r_N}y_{\beta_1}^{r_1}\cdots y_{\beta_N}^{r_N}\in K_\lambda\cap\mathcal{U}(\mathfrak{n}^-) = 0 \Longrightarrow \text{ all } \gamma_{r_1,\ldots,r_N} = 0$$

by the PBW Theorem on $\mathcal{U}(\mathfrak{n}^-)$. Thus, we have shown that we, in fact, have a basis of $M(\lambda)$ and so each element is uniquely expressible in the form um_λ for $u\in\mathcal{U}(\mathfrak{n}^-)$. □

9.6. **Definition.** For each 1-dimensional representation $\mu\colon\mathfrak{h}\to\mathbb{C}$, we define

$$M(\lambda)_\mu := \{m\in M(\lambda)\mid xm = \mu(x)m, \forall x\in\mathfrak{h}\}$$

9.7. **Proposition.** [Car05, Theorem 10.7] *When considered as an $\mathfrak{h}$-module, $M(\lambda)$ has the following properties.*

- *(a) $M(\lambda)_\mu$ is a subspace of $M(\lambda)$.*
- *(b) $M(\lambda) = \bigoplus_{\mu\in\mathfrak{h}^*}M(\lambda)_\mu$.*
- *(c) $M(\lambda)_\mu\neq 0$ if and only if $\lambda-\mu$ is a sum of positive roots.*
- *(d) $\dim M(\lambda)_\mu$ is equal to the number of ways of expressing $\lambda-\mu$ as a sum of positive roots.*

Proof. Part (a) follows from the definition of $M(\lambda)_\mu$ since it is closed under the action of $\mathfrak{h}$.

Since the elements $y_{\beta_1}^{r_1}\cdots y_{\beta_N}^{r_N}m_\lambda$ with $r_i\geq 0$ form a basis for $M(\lambda)$, part (b) follows by showing that

$$xy_{\beta_1}^{r_1}\cdots y_{\beta_N}^{r_N}m_\lambda = (\lambda - r_1\beta_1 - \cdots - r_N\beta_N)(x)y_{\beta_1}^{r_1}\cdots y_{\beta_N}^{r_N}m_\lambda, \forall x\in\mathfrak{h}$$

by induction on $r_1+\cdots+r_N$. Thus, $y_{\beta_1}^{r_1}\cdots y_{\beta_N}^{r_N}m_\lambda\in M(\lambda)_\mu$ for $\mu = \lambda - r_1\beta_1 - \cdots - r_N\beta_N$ and so

$$M(\lambda) = \sum_\mu M(\lambda)_\mu$$

In fact, this sum is direct by direct linear algebra computation by showing

$$M(\lambda)_\mu \cap (M(\lambda)_{\mu_1} + \cdots + M(\lambda)_{\mu_k}) = 0$$

for $\mu, \mu_1, \ldots, \mu_k \in \mathfrak{h}^*$ distinct.

Finally, take $\Lambda = \{\mu \in \mathfrak{h}^* \mid \lambda - \mu \text{ is a sum of positive roots}\}$. Then, for each $\lambda \in \Lambda$, let

$$N_\mu = \operatorname{span}\{y_{\beta_1}^{r_1} \cdots y_{\beta_N}^{r_N} m_\lambda \mid \lambda - r_1\beta_1 - \cdots - r_N\beta_N = \mu\} \subseteq M(\lambda)_\mu \subseteq M(\lambda)$$

By the PBW theorem, we also get

$$M(\lambda) = \bigoplus_{\mu \in \Lambda} N_\mu$$

However, since $M(\lambda) = \bigoplus_{\mu \in \mathfrak{h}^*} M(\lambda)_\mu$ from above, we get that $M(\lambda)_\mu = N_\mu$ for all $\mu \in \Lambda$ and $M(\lambda)_\mu = 0$ for $\mu \in \mathfrak{h}^*$ when $\mu \notin \Lambda$. Thus,

$$\dim M(\lambda)_\mu = \dim N_\mu = |\{(r_1, \ldots, r_N) \in \mathbb{N}_0^N \mid \lambda - r_1\beta_1 - \cdots - r_N\beta_N = \mu\}|$$

that is, the number of ways to write $\lambda - \mu$ as a sum of positive roots. ☐

9.8. **Definition.** $\mu \in \mathfrak{h}^*$ is a *weight* of $M(\lambda)$ if $M(\lambda)_\mu \neq 0$ and $M(\lambda)_\mu$ is called the *weight space* of $M(\lambda)$ with weight μ.

9.9. **Theorem.** [Car05, Theorem 10.9] *$M(\lambda)$ has a unique maximal submodule, denoted $J(\lambda)$.*

Proof. Take V to be a proper $\mathcal{U}(\mathfrak{g})$-submodule of $M(\lambda)$. The key idea is to show that $V \subseteq \sum_{\mu, \mu \neq \lambda} M(\lambda)_\mu \subsetneqq M(\lambda)$ and take $J(\lambda)$ to be the sum of all the proper submodules of $M(\lambda)$, thus yielding a unique maximal submodule of $M(\lambda)$ lying in this codimension 1 subspace.

By the theorem above,

$$v = \sum_{i=1}^{n} v_{\mu_i}, \quad v_{\mu_i} \in M(\lambda)_{\mu_i}$$

where the μ_i are distinct. To show each $v_{\mu_i} \in V$, we observe, for $x \in \mathfrak{h}$ and fixed i,

$$\begin{aligned} x v_{\mu_i} &= \mu_i(x) v_{\mu_i} \\ \implies \prod_{j, j \neq i} (x - \mu_j(x)) v &= \sum_k \prod_{j, j \neq i} (x - \mu_j(x)) v_{\mu_k} \\ &= \sum_k \prod_{j, j \neq i} (\mu_k(x) - \mu_j(x)) v_{\mu_k} \\ &= \prod_{j, j \neq i} (\mu_i(x) - \mu_j(x)) v_{\mu_i} \end{aligned}$$

Moreover, since any vector space cannot be a union of finitely many proper subspaces, we can find an $x \in \mathfrak{h}$ with $\mu_i(x) \neq \mu_j(x)$ for all $j \neq i$. Thus, if we pick such an x,

$$\prod_{j,j\neq i} (x - \mu_j(x))v \in V \Longrightarrow \prod_{j,j\neq i} (\mu_i(x) - \mu_j(x))v_{\mu_i} \in V$$

and so, since $\prod_{j.j\neq i}(\mu_i(x) - \mu_j(x)) \neq 0$, we get that $v_{\mu_i} \in V$.

If we define $V_\mu := V \cap M(\lambda)_\mu$, we need only show that $V_\lambda = V \cap M(\lambda)_\lambda = 0$. However, if this were not the case, $V_\lambda = M(\lambda)_\lambda$ since $\dim M(\lambda)_\lambda = 1$ so

$$m_\lambda \in V \Longrightarrow M(\lambda) = \mathcal{U}(\mathfrak{g})m_\lambda \subseteq V \Longrightarrow V = M(\lambda)$$

which is a contradiction. Thus, every proper submodule V of $M(\lambda)$ is contained in $\sum_{\mu,\mu\neq\lambda} M(\lambda)_\mu$. □

9.10. **Definition.** [Car05, Definition 10.10] For $\lambda, \mu \in \mathfrak{h}^*$, we say that $\lambda \succ \mu$ if $\lambda - \mu$ is a sum of positive roots. This defines a partial order on $\mathfrak{h}^*$.

By , we also know that the weights of $M(\lambda)$ are precisely the $\mu \in \mathfrak{h}^*$ with $\mu \prec \lambda$. Thus, λ is the highest weight of $M(\lambda)$ with respect to this partial order and so $M(\lambda)$ is called the *Verma module with highest weight* λ.

Finally, we define $L(\lambda) := M(\lambda)/J(\lambda)$.

9.11. **Proposition.**
(a) $L(\lambda)$ is an irreducible $\mathcal{U}(\mathfrak{g})$-module since $J(\lambda)$ is a maximal submodule of $M(\lambda)$.
(b) λ is a weight of $L(\lambda)$ since $J(\lambda)_\lambda = 0$.
(c) $\dim L(\lambda)_\lambda = 1$ since $J(\lambda)_\lambda = 0$ and λ is the highest weight of $L(\lambda)$.

Now that we have $L(\lambda)$, we still wish to figure out

(a) For which λ is $L(\lambda)$ finite dimensional?
(b) Which weights μ occur in $L(\lambda)$ and what are their multiplicities? In otherwords, what is $\dim L(\lambda)_\mu$?

9.12. **Theorem.** *Let V be a finite dimensional irreducible $\mathfrak{g}$-module with highest weight vector v_λ. Then, there exists a surjective homomorphism $\theta\colon M(\lambda) \to V$ of $\mathcal{U}(\mathfrak{g})$-modules such that $\theta(m_\lambda) = v_\lambda$.*

9.13. **Corollary.** *A finite dimensional irreducible $\mathfrak{g}$-module V with highest weight λ is isomorphic to $L(\lambda)$.*

Proof of Corollary. Consider the surjective homomorphism

$$\theta\colon M(\lambda) \twoheadrightarrow V$$

Since V is irreducible, $\ker\theta$ must be maximal in $M(\lambda)$, but $M(\lambda)$ has unique maximal submodule $J(\lambda)$. Therefore, $\ker\theta = J(\lambda)$ and so

$$L(\lambda) \cong M(\lambda)/J(\lambda) = M(\lambda)/\ker\theta \cong V$$

□

Note, however, the converse statement is not true. The partial converse is

9.14. **Proposition.** [Car05, Prop 10.15] *or* [Hum72, Thm 21.1] *If $L(\lambda)$ is finite dimensional, then $\lambda(h_i)$ is a non-negative integer for $\{h_1, \ldots, h_n\}$ a basis of $\mathfrak{h}$.*

Proof. Take simple root α_i corresponding to h_i and let $\mathfrak{sl}(\alpha_i) = \operatorname{span}\{x_{\alpha_i}, y_{\alpha_i}, h_i\}$ be the corresponding copy of $\mathfrak{sl}_2(\mathbb{C})$ in $\mathfrak{g}$. Since $L(\lambda)$ is thus also a finite-dimensional $\mathfrak{sl}(\alpha_i)$-module, there is a highest weight vector of weight λ, which is the weight for h_i. Thus, by virtue of our work on $\mathfrak{sl}_2$ (see 3.12), it must be that $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$. □

9.15. **Theorem.** [Car05, Thm 10.20] *Suppose $\lambda \in \mathfrak{h}^*$ is dominant and integral. Then, the irreducible $\mathfrak{g}$-module $L(\lambda)$ is finite dimensional and its set of weights is permuted by $\mathscr{W}$ with*

$$\dim L(\lambda)_\mu = \dim L(\lambda)_{w\mu} \quad \forall w \in \mathscr{W}$$

The proof of this theorem is quite long (see [Car05] or [Hum72, §21]), but the following lemma must be used in either proof.

9.16. **Lemma.** [Hum72, Lem 21.2] *The following identities hold in $\mathcal{U}(\mathfrak{g})$ for $k \geq 0, 1 \leq i, j \leq \ell$.*

(a) $[x_j, y_i^{k+1}] = 0$ *when* $i \neq j$
(b) $[h_j, y_i^{k+1}] = -(k+1)\alpha_i(h_j) y_i^{k+1}$
(c) $[x_i, y_i^{k+1}] = -(k+1) y_i^k (k \cdot 1 - h_i)$

Proof of Lemma. (a) $\alpha_j - \alpha_i$ is not a root, so the result follows. why?

(b) When $k = 0$,

$$[h_j, y_i] = -\alpha_i(h_j) y_i$$

Now, asume the result holds for all values less than $k+1$. We get

$$\begin{aligned}
[h_j, y_i^{k+1}] &= h_j y_i^{k+1} - y_i^{k+1} h_j \\
&= (h_j y_i^k - y_i^k h_j) y_i + y_i^k (h_j y_i - y_i h_j) \\
&= [h_j, y_i^k] y_i + y_i^k [h_j, y_i] \\
&= -k\alpha_i(h_j) y_i^k y_i + y_i^k(-\alpha_i(h_j) y_i) \\
&= -(k+1)\alpha_i(h_j) y_i^{k+1}
\end{aligned}$$

(c) Similar to the above, we check

$$\begin{aligned}
[x_i, y_i^{k+1}] &= x_i y_i^{k+1} - y_i^{k+1} x_i \\
&= [x_i, y_i] y_i^k + y_i [x_i, y_i^k] \\
&= h_i y_i^k + y_i(-k y_i^{k-1}((k-1) \cdot 1 - h_i)) \\
&= h_i y_i^k - y_i^k h_i + (k+1) y_i^k h_i - k y_i^k (k-1) \\
&= -k\alpha_i(h_i) y_i^k + (k+1) y_i^k h_i - k y_i^k (k-1)
\end{aligned}$$

$$= -2ky_i^k + (k+1)y_i^k h_i - ky_i^k(k-1)$$
$$= (k+1)y_i^k h_i - k(k+1)y_i^k$$
$$= -(k+1)y_i^k(k\cdot 1 - h_i)$$

□

10. Characters

10.1. **Definition.** Given $\mathfrak{g}$ a semisimple Lie algebra and V be a $\mathfrak{g}$-module with weight space decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

where $\dim V_\lambda < \infty$. Then, we define the *character of V* to be the function $\operatorname{ch} V \colon \mathfrak{h}^* \to \mathbb{Z}$ given by

$$(\operatorname{ch} V)(\lambda) = \dim V_\lambda$$

and we define the *formal character of V* to be

$$\operatorname{ch} V = \sum_{\mu} \dim V_\mu e_\mu \in \mathbb{Z}[\Lambda_W]$$

where the sum is over all weights of V and $e_\mu(\lambda) = \delta_{\mu\lambda}$.

Explain the ring structure here; the multiplication is not clear.

10.2. **Proposition.** *Let V, V_1, V_2 be $\mathfrak{g}$-modules that admit characters. Then,*

(a) If $U \subseteq V$,

$$\operatorname{ch} U + \operatorname{ch}(V/U) = \operatorname{ch} V$$

(b) Therefore,

$$\operatorname{ch}(V_1 \oplus V_2) = \operatorname{ch} V_1 + \operatorname{ch} V_2$$

(c)

$$\operatorname{ch}(V_1 \otimes V_2) = \operatorname{ch} V_1 \cdot \operatorname{ch} V_2$$

10.3. **Lemma.** *Given a Verma module $M(\lambda)$ with $\lambda \in \mathfrak{h}^*$, we have that*

$(\operatorname{ch} M(\lambda))(\mu) =$ *Number of ways to write $\lambda - \mu$ as a positive sum of positive roots.*

Proof. This follows immediately from the definition of $\operatorname{ch} M(\lambda)$ and 9.7(d). □

10.4. **Proposition.** *If $\mathfrak{P}(\nu)$ is the number of ways to write ν as a positive sum of positive roots, then*

$$\operatorname{ch} M(\lambda) = e_\lambda \left(\sum_{\nu \in \mathfrak{h}^*} \mathfrak{P}(\nu) e_{-\nu} \right)$$

Proof. This follows since

$$\begin{aligned}
\operatorname{ch} M(\lambda) &= \sum_{\mu \in \mathfrak{h}^*} \mathfrak{P}(\lambda - \mu) e_\mu \\
&= \sum_{\nu \in \mathfrak{h}^*} \mathfrak{P}(\nu) e_{\lambda - \nu} \\
&= \sum_{\nu \in \mathfrak{h}^*} \mathfrak{P}(\nu) e_\lambda e_{-\nu} \\
&= e_\lambda \sum_{\nu \in \mathfrak{h}^*} \mathfrak{P}(\nu) e_{-\nu}
\end{aligned}$$

□

10.5. **Lemma.** *In the ring of all function $f \colon \mathfrak{h}^* \to \mathbb{Z}$ with finite support,*

$$\left(\sum_{\nu \in \mathfrak{h}^*} \mathfrak{P}(\nu) e_{-\nu} \right)^{-1} = \prod_{\alpha \in \Phi^+} (1 - e_{-\alpha})$$

Proof. Let $\Phi^+ = \{\beta_1, \dots, \beta_n\}$. Then,

$$\begin{aligned}
\sum_{\nu} \mathfrak{P}(\nu) e_{-\nu} &= \sum_{r_1, \dots, r_n \geq 0} e_{-r_1 \beta_1 - \cdots - r_n \beta_n} \\
&= \sum_{r_1, \dots, r_n \geq 0} e_{-\beta_1}^{r_1} \cdots e_{-\beta_n}^{r_n} \\
&= \prod_{i=1}^{n} \left(\sum_{r_1 \geq 0} e_{-\beta_i}^{r_i} \right)
\end{aligned}$$

but then

$$(1 - e_{-\beta_i})(1 + e_{-\beta_i} + e_{-\beta_i}^2 + \cdots) = 1$$

and so we have our desired inverse

$$\prod_{i=1}^{n} (1 - e_{-\beta_i}) = \prod_{\alpha \in \Phi^+} (1 - e_{-\alpha})$$

□

We state the following theorem without proof.

10.6. **Theorem.** *The character of the Verma module with highest weight λ is given by*

$$\operatorname{ch} M(\lambda) = \frac{e_{\lambda + \rho}}{e_\rho \prod_{\alpha \in \Phi^+} (1 - e_{-\alpha})}$$

Proof.

$$\operatorname{ch} M(\lambda) = e_\lambda \left(\sum_{\nu \in \mathfrak{h}^*} \mathfrak{P}(\nu) e_{-\nu} \right) = \frac{e_\lambda}{\prod_{\alpha \in \Phi^+} (1 - e_{-\alpha})} = \frac{e_{\lambda + \rho}}{e_\rho \prod_{\alpha \in \Phi^+} (1 - e_{-\alpha})}$$

$\square$

10.7. **Theorem.** [Car05, Theorem 12.16] *The Verma module $M(\lambda)$ has a finite composition series*

$$M(\lambda) = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_r = 0$$

where each N_i is a submodule of $M(\lambda)$ and N_{i+1} is a maximal submodule of N_i. Furthermore, $N_i/N_{i+1} \cong L(w \cdot \lambda)$ for some $w \in \mathscr{W}$.

10.8. **Theorem** (Weyl Character Formula)**.** [Car05, Theorem 12.17] *Given semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$. Then, for λ a dominant weight,*

$$\operatorname{ch} L(\lambda) = \frac{\sum_{w\in\mathscr{W}} (-1)^{\ell(w)} e_{w(\lambda+\rho)}}{\sum_{w\in\mathscr{W}} (-1)^{\ell(w)} e_{w(\rho)}}$$

Proof. Since λ is dominant and integral, $\lambda(h_i) \geq 0$ for all h_i. Therefore,

$(\lambda+\rho)(h_i) = \lambda(h_i)+1 > 0 \implies \lambda+\rho$ is in the fundamental Weyl chamber, C

Thus, $w(\lambda+\rho) \in w(C)$ and $w(\lambda+\rho)$ for each $w \in \mathscr{W}$.

The highest weight for $M(w\cdot\lambda)$ is $w\cdot\lambda = w(\lambda+\rho)-\rho$ and so the characters $\operatorname{ch} M(w\cdot\lambda)$ are linearly independent for $w \in \mathscr{W}$ and similarly for $L(w\cdot\lambda)$.

Let $M(w\cdot\lambda)$ have composition factors $L(y\cdot\lambda)$ for some $y \in \mathscr{W}$. Then, since $w\cdot\lambda$ is the highest weight, it must be that $y\cdot\lambda \prec w\cdot\lambda$ and $\dim(M(w\cdot\lambda)_{w\cdot\lambda}) = 1 \implies \dim(L(w\cdot\lambda)_{w\cdot\lambda}) = 1$. Thus, we have

$$\operatorname{ch} M(w\cdot\lambda) = \operatorname{ch} L(w\cdot\lambda) + \sum_{y\in\mathscr{W}, y\cdot\lambda\prec w\cdot\lambda} a_{w,y} \operatorname{ch} L(y\cdot\lambda)$$

where $a_{w,y} \in \mathbb{N}_{>0}$, so the characters are unitriangularily related, meaning we can create an inverse, giving

$$\operatorname{ch} L(w\cdot\lambda) = \operatorname{ch} M(w\cdot\lambda) + \sum_{y\in\mathscr{W}, y\neq w} b_{w,y} \operatorname{ch} M(y\cdot\lambda)$$

with $b_{w,y} \in \mathbb{Z}$ (not necessarily non-negative). If we specify $w=1$, then we get

$$\operatorname{ch} L(\lambda) = \sum_{y\in\mathscr{W}} b_{1,y} \operatorname{ch} M(y\cdot\lambda) = \sum_{y\in\mathscr{W}} b_{1,y} \frac{e_{y(\lambda+\rho)}}{e_\rho \prod_{\alpha\in\Phi^+}(1-e_{-\alpha})}$$

Now, by 9.15,

$$\dim L(\lambda)_\mu = \dim L(\lambda)_{w(\mu)} \implies w(\operatorname{ch} L(\lambda)) = \operatorname{ch} L(\lambda)$$

for all $w \in \mathscr{W}$. Also,

$$s_i\left(e_\rho \prod_{\alpha\in\Phi^+}(1-e_{-\alpha})\right) = s_i\left(\sum_{w\in\mathscr{W}} (-1)^{\ell(w)} e_{w(\rho)}\right) = -\sum_{w\in\mathscr{W}} (-1)^{\ell(w)} e_{w(\rho)}$$

and so

$$w\left(e_\rho \prod_{\alpha\in\Phi^+}(1-e_{-\alpha})\right) = (-1)^{\ell(w)} e_\rho \prod_{\alpha\in\Phi^+}(1-e_{-\alpha})$$

Therefore,

$$\operatorname{ch} L(\lambda) = w(\operatorname{ch} L(\lambda)) = w\left(\sum_{y\in\mathscr{W}} b_{1,y}\frac{e_{y(\lambda+\rho)}}{e_\rho\prod_{\alpha\in\Phi^+}(1-e_{-\alpha})}\right) = (-1)^{\ell(w)}\sum_{y\in\mathscr{W}} b_{1,y}\frac{w(e_{y(\lambda+\rho)})}{e_\rho\prod_{\alpha\in\Phi^+}(1-e_{-\alpha})}$$

tells us that

$$w\left(\sum_{y\in W} b_{1,y}e_{y(\lambda+\rho)}\right) = (-1)^{\ell(w)}\left(\sum_{y\in W} b_{1,y}e_{y(\lambda+\rho)}\right)$$

and since $we_\lambda = e_{w\lambda}$, we get $b_{1,w^{-1}y} = (-1)^{\ell(w)}b_{1,y}$ from the linear independence of the e_λ's. In particular, $b_{1,w^{-1}} = (-1)^{\ell(w)}$ since $b_{1,1} = 1$ and so $b_{1,w} = (-1)^{\ell(w^{-1})} = (-1)^{\ell(w)}$. Thus, since this is true for every $w \in \mathscr{W}$, we get

$$\operatorname{ch} L(\lambda) = \frac{\sum_{w\in\mathscr{W}}(-1)^{\ell(w)}e_{w(\lambda+\rho)}}{e_\rho\prod_{\alpha\in\Phi^+}(1-e_{-\alpha})} = \frac{\sum_{w\in\mathscr{W}}(-1)^{\ell(w)}e_{w(\lambda+\rho)}}{\sum_{w\in\mathscr{W}}(-1)^{\ell(w)}e_{w(\rho)}}$$

□

Note that, for $\lambda = 0$,

$$\operatorname{ch} L(0) = e_0 \Longrightarrow \sum_{w\in\mathscr{W}}(-1)^{\ell(w)}e_{w(\rho)} = \left(e_\rho\prod_{\alpha\in\Phi^+}(1-e_{-\alpha})\right)$$

so the Weyl Denominator Formula is a special case of the Weyl Character Formula (although we used the Weyl Denominator Formula in our proof of the Weyl Character Formula).

10.9. **Example.** Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Then, $\mathscr{W} = \mathfrak{S}_2$ and $\rho = 1$. So, for $n \in \mathbb{Z}_{\geq 0}$,

$$\operatorname{ch} L(n) = \frac{(-1)^0e_{n+1}+(-1)^1e_{-(n+1)}}{(-1)^0e_1+(-1)^1e_{-1}} = \frac{e_{n+1}-e_{-n-1}}{e_1-e_{-1}} = e_n+e_{n-2}+\cdots+e_{-n}$$

recovering our knowledge for finite dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules.

10.10. **Theorem** (Kostant's Multiplicity Formula)**.** *Let λ be a dominant weight and μ a weight. Then,*

$$\dim L(\lambda)_\mu = \sum_{w\in\mathscr{W}}(-1)^{\ell(w)}\mathfrak{P}(w(\lambda+\rho)-(\mu+\rho))$$

10.11. **Theorem** (Weyl's dimension formula)**.** *Let λ be a dominant weight. Then,*

$$\dim L(\lambda) = \frac{\prod_{\alpha\in\Phi^+}\langle\lambda+\rho,\alpha\rangle}{\prod_{\alpha\in\Phi^+}\langle\rho,\alpha\rangle}$$

10.12. **Example.** Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Then, since $\Phi^+ = \{1\}$ and $\rho = 1$,

$$\dim L(n) = \frac{\langle n+1, 1\rangle}{\langle 1, 1\rangle} = n+1$$

once again recovering knowledge from previous sections.

10.13. **Example.** More generally, for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, let $\lambda = \sum_{k=1}^{n-1} \lambda_k \epsilon_k$. Then,

$$\rho = \omega_1 + \omega_2 + \cdots + \omega_{n-1} = \epsilon_1 + (\epsilon_1 + \epsilon_2) + \cdots + (\epsilon_1 + \cdots + \epsilon_{n-1}) = \sum_{k=1}^{n-1} (n-k)\epsilon_k$$

and so

$$\begin{aligned}
\dim L(\lambda) &= \prod_{\alpha \in \Phi^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \\
&= \prod_{1 \le i < j \le n} \frac{\langle \sum_{k=1}^{n-1} (\lambda_k + n - k)\epsilon_k, \epsilon_i - \epsilon_j \rangle}{\langle \sum_{n=1}^{k} (n-k)\epsilon_k, \epsilon_i - \epsilon_j \rangle} \\
&= \prod_{1 \le i < j \le n} \frac{(\lambda_i + n - i) - (\lambda_j + n - j)}{(n-i) - (n-j)} \\
&= \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i}
\end{aligned}$$

In fact, the formula above is given by specializing the "Schur function" $s_\lambda(x_1 \ldots, x_n)$ to $s_\lambda(1, \ldots, 1)$. Furthermore, since

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathrm{SSYT}_n(\lambda)} x^{\mathrm{weight}(T)}$$

where $\mathrm{SSYT}_n(\lambda)$ is the set of all semistandard Young tableaux with letters in $\{1, \ldots, n\}$, we get

$$\dim L(\lambda) = s_\lambda(1, \ldots, 1) = |\,\mathrm{SSYT}_n(\lambda)|$$

10.14. **Remark.** In fact, in type A, one can see the Weyl Character Formula gives a "Schur function" using the Jacobi Bialternate Formula characterization of Schur functions. In type A, $\mathscr{W} \cong \mathfrak{S}_n$ and $\Lambda_W = \mathrm{span}_{\mathbb{Z}}\{\epsilon_1, \ldots, \epsilon_n\}$.

Then, since $e_\lambda = e_{\lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n}$, we get

$$\sum_{w \in \mathfrak{S}_n} (-1)^{\ell(w)} e_{w(\lambda+\rho)} = \sum_{w \in \mathfrak{S}_n} \mathrm{sgn}(w) \prod_{i=1}^{n} e_{(\lambda_i + n - i)\epsilon_{w(i)}} = \det \begin{pmatrix} e_{(\lambda_1+n-1)\epsilon_1} & e_{(\lambda_1+n-1)\epsilon_2} & \cdots & e_{(\lambda_1+n-1)\epsilon_n} \\ e_{(\lambda_2+n-2)\epsilon_1} & e_{(\lambda_2+n-2)\epsilon_2} & \cdots & e_{(\lambda_2+n-2)\epsilon_n} \\ \vdots & \vdots & & \vdots \\ e_{(\lambda_n)\epsilon_1} & e_{(\lambda_n)\epsilon_2} & \cdots & e_{(\lambda_n)\epsilon_n} \end{pmatrix}$$

Then, the initiated observe that this is the Vandermonde determinant $\Delta_{\lambda+\rho}$ where $e_{(\lambda_i+n-i)\epsilon_j}$ corresponds to the variable $x_j^{\lambda_i+n-1}$. Thus, in our type A setting,

$$\operatorname{ch} L(\lambda) = \frac{\sum_{w \in \mathfrak{S}_n} (-1)^{\ell(w)} e_{w(\lambda+\rho)}}{\sum_{w \in \mathfrak{S}_n} (-1)^{\ell(w)} e_{w(\rho)}} = \frac{\Delta_{\lambda+\rho}}{\Delta_\rho} = s_\lambda(e_{\epsilon_1}, e_{\epsilon_2}, \ldots, e_{\epsilon_n})$$

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