REPRESENTATION THEORY OF SYMMETRIC GROUPS

GEORGE H. SEELINGER

1. Introduction

The representation theory of symmetric groups is a well-studied and rich subject with connections to the representation theory of Lie groups and Lie algebras, as well as to symmetric function theory and combinatorics.

This monograph will assume the reader is already familiar with material in [See17, Sections 1–14] and [See18, Section 2], although not all of it is strictly speaking necessary. In this monograph, we will follow the program in [FH91].

Our results are all stated over $\mathbb{C}$ unless otherwise noted. $\mathfrak{S}_d$ is a symmetric group on d letters.

2. Small Examples

For small symmetric groups, one can use the theory of the representation theory of finite groups to directly compute the character tables of $\mathfrak{S}_n$. For all symmetric groups, we have the trivial representation and the sign representation given by $w.v = \operatorname{sgn}(w)v$ for $w \in \mathfrak{S}_n$.

2.1. **Example.** For $G = \mathfrak{S}_3$, since there are 3 conjugacy classes, there is only one missing representation of dimension 2. Thus, giving the remaining character table values by using the character orthogonality relations.

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
θ	2	0	-1

3. Characters of Symmetric Groups Representations

In this section, we follow the program of [Man98, Section 1.6] to develop some general character theory for $\mathfrak{S}_n$. Let $R^{(n)}$ be the free $\mathbb{Z}$-module generated by the irreducible characters of $\mathfrak{S}_n$ with $R^{(0)} = \mathbb{Z}$.

3.1. **Proposition.** *The direct sum*

$$R = \bigoplus_{n \geq 0} R^{(n)}$$

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has the structure of an associative and commutative graded ring under the product, for $\phi \in R^{(m)}$ and $\psi \in R^{(n)}$,

$$\phi \cdot \psi = \mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} (\phi \times \psi)$$

3.2. **Definition.** For $w \in \mathfrak{S}_n$, let $\lambda(w)$ be the partition of size n encoding the cycle type of w. Then, the *characteristic map* $\mathrm{ch}\colon R \to \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ is defined by, for $\phi \in R^{(n)}$,

$$\mathrm{ch}(\phi) := \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \phi(w) p_{\lambda(w)}$$

where $p_{\lambda(w)}$ is the power sum symmetric function.

3.3. **Theorem.** [Man98, Proposition 1.6.3] *The characteristic map defines a graded ring isomorphism from the ring R of the characters of the symmetric group to the ring Λ of symmetric functions.*

3.4. **Lemma.**

$$\mathrm{ch}(\phi) = \sum_{|\lambda|=n} z_\lambda^{-1} \phi_\lambda p_\lambda$$

where ϕ_λ is the value of ϕ on the conjugacy class of cycle type λ and z_λ is the cardinality of the centralizer of an element associated to the conjugacy class associated to λ, that is, $z_\lambda = \prod_i i^{m_i} m_i!$ where m_i is the multiplicity of i in λ.

Proof of Lemma. First we break up the sum

$$\sum_{w \in \mathfrak{S}_n} \phi(w) p_{\lambda(w)} = \sum_{|\lambda|=n} \sum_{w \text{ of cycle type } \lambda} \phi(w) p_\lambda$$

and, since characters are class functions, we may define ϕ_λ as $\phi(w)$ for any w with cycle type λ. Finally, the size of the conjugacy class must be $\frac{n!}{z_\lambda}$ by the orbit-stabilizer theorem, so we get

$$\sum_{|\lambda|=n} \sum_{w \text{ of cycle type } \lambda} \phi(w) p_\lambda = \sum_{|\lambda|=n} \frac{n!}{z_\lambda} \phi_\lambda p_\lambda$$

giving us the desired formula after multiplying both sides by $n!$. □

3.5. **Lemma.** ch *is an isometry, that is*

$$(\phi, \psi) = \langle \mathrm{ch}(\phi), \mathrm{ch}(\psi) \rangle$$

where $(\cdot,\cdot)$ is the inner product on characters and $\langle \cdot, \cdot \rangle$ is the Hall-inner product on symmetric functions. In particular, this means ch *is injective.*

Proof of Lemma. We check, for $\phi, \psi \in R^{(n)}$,

$$\begin{aligned} \langle \mathrm{ch}(\phi), \mathrm{ch}(\psi) \rangle &= \langle \sum_{\lambda \vdash n} z_\lambda^{-1} \phi_\lambda p_\lambda, \sum_{\mu \vdash n} z_\mu^{-1} \psi_\mu p_\mu \rangle \\ &= \sum_{\lambda \vdash n} \sum_{\mu \vdash n} \phi_\lambda \psi_\mu z_\lambda^{-1} z_\mu^{-1} \langle p_\lambda, p_\mu \rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\lambda \vdash n} \sum_{\mu \vdash n} \phi_\lambda \psi_\mu z_\lambda^{-1} z_\mu^{-1} z_\lambda \delta_{\lambda,\mu} \\
&= \sum_{\lambda \vdash n} \phi_\lambda \psi_\lambda z_\lambda^{-1} \\
&= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \phi(w)\psi(w) \\
&= (\phi, \psi)
\end{aligned}
$$

□

Proof of Theorem. First, we must define the class function $p\colon \mathfrak{S}_n \to \Lambda^n$ via

$$p(w) = p_{\lambda(w)}$$

Then, we can rephrase

$$\mathrm{ch}(\phi) = (\phi, p)$$

We check that

$$
\begin{aligned}
\mathrm{ch}(\phi \cdot \psi) &= (\phi \cdot \psi, p) \\
&= (\mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} (\phi \times \psi), p) \\
&= (\phi \times \psi, \mathrm{Res}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} p) && \text{by Frobenius Reciprocity} \\
&= \frac{1}{m!n!} \sum_{(w,w') \in \mathfrak{S}_m \times \mathfrak{S}_n} (\phi \times \psi)(ww') \overline{p(ww')} && \text{by definition of } (\cdot,\cdot) \\
&= \frac{1}{m!n!} \sum_{w \in \mathfrak{S}_m, w' \times \mathfrak{S}_n} \phi(w)\psi(w') p_w p_{w'} \\
&= \left(\frac{1}{m!} \sum_{w \in \mathfrak{S}_m} \phi(w) p_w \right) \left(\frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \psi(w) p_w \right) \\
&= \mathrm{ch}(\phi)\,\mathrm{ch}(\psi)
\end{aligned}
$$

Now, consider the trivial character $1_n \in R^{(n)}$ of $\mathfrak{S}_n$. We compute

$$\mathrm{ch}(1_n) = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda = h_n$$

where the h_n is the homogeneous symmetric polynomial and the equality comes from an argument on generating functions (see [See18, Section 2]). Furthermore, since Λ is algebraically generated by $\{h_n\}_{n \in \mathbb{N}}$, it must be that Λ is in the image of ch. Furthermore, since ch is also injective, it must be that ch is an isomorphism. □

It bears repeating from the proof above.

3.6. **Corollary** (Corollary of proof). $\mathrm{ch}(1_n) = h_n$ *for* 1_n *the irreducible character of the trivial representation of* $\mathfrak{S}_n$.

3.7. **Proposition.** *We have that, under the characteristic map, the elementary functions e_n correspond to the character of the sign representation of $\mathfrak{S}_n$, say ϵ.*

Proof. By our alternate characterization of the characteristic map,

$$\mathrm{ch}(\epsilon) = \sum_{\lambda \vdash n} z_\lambda^{-1} \epsilon(\lambda) p_\lambda = e_n$$

where the last equality follows from an argument on generating functions for p_n and e_n (see [See18]). $\square$

3.8. **Proposition.** *The irreducible characters of $\mathfrak{S}_n$ are given by $\{\mathrm{ch}^{-1}(s_\lambda) \mid \lambda \vdash d\}$.*

Proof. Recall that the irreducible characters of a group G form an orthonormal basis for the set of class functions of G under the inner product $(\cdot, \cdot)$, and since the set of class functions is a $\mathbb{Z}$-module, this basis is unique. Since ch is an isometry and the Schur functions s_λ form an orthonormal basis of Λ under the Hall-inner product, it must be that $\{\mathrm{ch}^{-1}(s_\lambda) \mid \lambda \vdash n\}$ is the set of all irreducible characters of $\mathfrak{S}_n$ up to sign. We will later show they are all positive when evaluated on $1 \in \mathfrak{S}_n$. $\square$

3.9. **Definition.** We will denote the irreducible character $\chi_\lambda := \mathrm{ch}^{-1}(s_\lambda)$.

3.10. **Proposition.** $\chi_\lambda = \det(1_{\lambda_i - i + j})_{1 \le i,j \le n}$ *where* $1_{\lambda_i - i+j}$ *is the trivial character for* $\mathfrak{S}_{\lambda_i - i + j}$ *(and* 0 *if* $\lambda_i - i + j \le 0$*).*

Proof. The Jacobi-Trudi identity tells us that, for $\lambda \vdash n$,

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \le i,j \le n}$$

From above, we have $\mathrm{ch}(1_n) = h_n$ and so, apply ch^{-1} to both sides, we get our result. $\square$

3.11. **Theorem** (Frobenius Character Formula)**.** [Man98, 1.6.6] *Given a partition $\mu \vdash n$,*

$$p_\mu = \sum_{\lambda \vdash n} \chi_\lambda(\mu) s_\lambda$$

where $\chi_\lambda(\mu) = \chi_\lambda(w)$ for $w \in \mathfrak{S}_n$ of cycle type μ.

Proof. First, we observe that $\mathrm{ch}^{-1}(p_\mu) = z_\mu f_\mu$ where

$$f_\mu(w) = \begin{cases} 1 & \text{if } w \text{ has cycle type } \mu \\ 0 & \text{else} \end{cases}$$

Next, by the fact that ch is an isometry,

$$\langle s_\lambda, p_\mu \rangle = (\chi_\lambda, z_\mu f_\mu) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \chi_\lambda(w) z_\mu f_\mu(w) = \frac{z_\mu}{n!} \sum_{w \text{ with cycle type } \mu} \chi_\lambda(w) = \chi_\lambda(\mu)$$

since the size of the conjugacy class is $\frac{n!}{z_\mu}$. Therefore, for $1 \in \mathfrak{S}_n$,

$$\chi_\lambda(1) = \langle s_\lambda, p_{1^n} \rangle = \langle s_\lambda, h_{1^n} \rangle = K_{\lambda,1^n} > 0$$

since $K_{\lambda,1^n}$ is the number of standard tableaux of shape λ. $\square$

3.12. **Corollary** (Corollary of proof)**.** [Man98, Corollary 1.6.8] *The dimension of the irreducible representation of $\mathfrak{S}_n$ with character χ_λ is equal to the number of standard tableaux of shape λ.*

3.13. **Corollary.** *We can invert the Frobenius character formula to get*

$$s_\lambda = \sum_{\mu \vdash n} z_\mu^{-1} \chi_\lambda(\mu) p_\mu$$

Proof. We know from our arguments proving the Frobenius Character Formula that

$$s_\lambda = \operatorname{ch}(\chi_\lambda) = \sum_{\mu \vdash n} z_\mu^{-1} \chi_\lambda(\mu) p_\mu$$

where the second equality follows from our alternate characterization of the characteristic map. $\square$

4. Explicitly Constructing Representations

Given our knowledge of character theory above, let us systematically construct some representations.

4.1. **Definition.** Given a vector space V let $\mathfrak{S}_d$ act on $V^{\otimes d} = V \otimes \cdots \otimes V$ by permuting the terms of the tensor product. In other words, for $v_1, v_2, \ldots, v_n \in V$ (not necessarily distinct), let

$$w.(v_1 \otimes \cdots \otimes v_n) = v_{w(1)} \otimes \cdots \otimes v_{w(n)}$$

Given the symmetric group action defined above, we can also induce the action on $\operatorname{Sym}^r V$ and $\wedge^r V$.

4.2. **Proposition.** *Given the action of $\mathfrak{S}_r$ on $V^{\otimes r}$, we get that*

- *(a)* $\operatorname{Sym}^r V$ *is the trivial representation with character h_r under the characteristic map.*
- *(b)* $\wedge^r V$ *is the sign representation with character e_r under the characteristic map.*

Proof. First, consider $\operatorname{Sym}^r V$ as a representation of $\mathfrak{S}_r$. Then, any $w \in \mathfrak{S}_r$ permutes the terms of $v_1 \otimes \cdots \otimes v_d$, but this yields the same element by definition of the symmetric power. Thus, this must be the trivial representation of $\mathfrak{S}_r$ with character h_r.

Similarly, if we consider $\wedge^r V$ as a representation of $\mathfrak{S}_r$, $w \in \mathfrak{S}_r$ permutes the terms of $v_1 \wedge \cdots \wedge v_r$, but then

$$v_{w(1)} \wedge \cdots \wedge v_{w(r)} = \operatorname{sign}(w)(v_1 \wedge \cdots \wedge v_r)$$

by definition of the exterior power. Thus, we get that $\wedge^r V$ is the sign representation of $\mathfrak{S}_r$ with character e_r. $\square$

4.3. **Corollary.** *Given* $\mathfrak{S}_{r_1} \times \cdots \times \mathfrak{S}_{r_\ell}$*, then*

$$\mathrm{Sym}^{r_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{r_\ell}(V)$$

is the trivial representation and

$$\wedge^{r_1}(V) \otimes \cdots \otimes \wedge^{r_\ell}(V)$$

is the sign representation.

Proof. The first assertion follows immediately from the action of the group on this symmetric power; the action must be trivial. Similarly, from the above, it is almost immediate that

$$\begin{aligned} &(w_1, \ldots, w_r).(u_1 \wedge \cdots \wedge u_{r_1} \otimes v_1 \wedge \cdots \wedge v_{r_2} \otimes \cdots \otimes w_1 \wedge \cdots \wedge w_{r_\ell}) \\ = &(\mathrm{sign}(w_1)\,\mathrm{sign}(w_2) \cdots \mathrm{sign}(w_r))(u_1 \wedge \cdots \wedge u_{r_1} \otimes v_1 \wedge \cdots \wedge v_{r_2} \otimes \cdots \otimes w_1 \wedge \cdots \wedge w_{r_\ell}) \end{aligned}$$

$\square$

4.4. **Definition.** For a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_\ell}$. Then, we define induced modules

$$H_\lambda := \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_r}(\rho_1) \quad E_{\lambda'} := \mathrm{Ind}_{\mathfrak{S}_{\lambda'}}^{\mathfrak{S}_r}(\rho_{sign})$$

where ρ_1 is the trivial representation and ρ_{sign} is the sign representation.

4.5. **Proposition.** *Given a partition* λ*, the characterstic map applied to the character of* H_λ *gives* h_λ *and the characteristic map applied to the character of* $E_{\lambda'}$ *gives* $e_{\lambda'}$*.*

Proof. Let χ_{λ_i} be the character of the trivial representation for $\mathfrak{S}_{\lambda_i}$. Then, consider that the character of H_λ is $\mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_r}(\chi_{\lambda_1} \times \cdots \times \chi_{\lambda_\ell}) = \chi_{\lambda_1} \cdots \chi_{\lambda_\ell}$. Thus, since the characteristic map is a ring isomorphism,

$$\mathrm{ch}(\chi_{\lambda_1} \cdots \chi_{\lambda_\ell}) = \mathrm{ch}(\chi_{\lambda_1}) \cdots \mathrm{ch}(\chi_{\lambda_\ell}) = h_{\lambda_1} \cdots h_{\lambda_\ell} = h_\lambda$$

A nearly identical argument gives the result for $E_{\lambda'}$. $\square$

5. Young Symmetrizers and Specht Modules

The Frobenius character formula suggests that we will need to have the symmetric group act on polynomials associated to standard tableaux in order to explicitly realize the irreducible representations of $\mathfrak{S}_n$. There are a few ways to do this, one of which we expand on below, following [FH91].

5.1. **Definition.** Given a tableau T of shape λ labelled with integers $1, \ldots, d$, we define subgroups of $\mathfrak{S}_d$

$$R_{\mathsf{T}} := \{w \in \mathfrak{S}_d \mid w \text{ preserves each row of } \mathsf{T}\}$$

and

$$C_{\mathsf{T}} := \{w \in \mathfrak{S}_d \mid w \text{ preserves each column of } \mathsf{T}\}$$

Furthermore, we define elements of $\mathbb{C}\mathfrak{S}_d$, the *row stabalizer*

$$a_{\mathsf{T}} := \sum_{w \in R_{\mathsf{T}}} e_w$$

and the *column stabalizer*

$$b_{\mathsf{T}} := \sum_{w \in C_{\mathsf{T}}} \operatorname{sgn}(w) e_w$$

If T^* is the canonical standard tableau of shape λ, we define

$$R_\lambda := R_{\mathsf{T}^*}, C_\lambda := C_{\mathsf{T}^*}, a_\lambda := a_{\mathsf{T}^*}, b_\lambda := b_{\mathsf{T}^*}$$

5.2. **Proposition.** *Given that action of $\mathfrak{S}_d$ on $V^{\otimes d}$ via*

$$w.(v_1 \otimes \cdots \otimes v_d) = v_{w(1)} \otimes \cdots \otimes v_{w(d)}$$

that is, w permutes the terms in $v_1 \otimes \cdots \otimes v_d$, we observe

(a)

$$\operatorname{im}(a_\lambda) = \operatorname{Sym}^{\lambda_1} V \otimes \operatorname{Sym}^{\lambda_2} V \otimes \cdots \otimes \operatorname{Sym}^{\lambda_\ell} V$$

(b)

$$\operatorname{im}(b_\lambda) = \wedge^{\lambda'_1} V \otimes \wedge^{\lambda'_2} V \otimes \cdots \otimes \wedge^{\lambda'_k} V$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_k)$ is the conjugate partition to λ.

Proof. We observe, for $p \in R_\lambda$

$$p \cdot a_\lambda = a_\lambda \cdot p = a_\lambda$$

which follows almost immediately. Similarly, for $q \in C_\lambda$, we have

$$q \cdot b_\lambda = b_\lambda \cdot q = b_\lambda$$

□

5.3. **Definition.** We define the *Young symmetrizer* to be the element

$$c_\lambda := a_\lambda b_\lambda$$

5.4. **Example.** If $\lambda = (d)$, then

$$c_{(d)} = a_{(d)} = \sum_{w \in \mathfrak{S}_d} e_w$$

and when $\lambda = (1, \ldots, 1)$, then

$$c_{(1,\ldots,1)} = b_{(1,\ldots,1)} = \sum_{w \in \mathfrak{S}_d} \operatorname{sgn}(w) e_w$$

Finally, for $\lambda = (2, 1)$, we have

$$c_{(2,1)} = (e_1 + e_{(12)})(e_1 - e_{(13)}) = 1 + e_{(12)} - e_{(13)} - e_{(132)}$$

We will compute many other examples as needed.

5.5. **Proposition.** *The set* $\{c_\lambda\}_{\lambda\vdash d}$ *form a set of seminormal idempotents up to a scalar. That is,*

$$c_\lambda c_\mu = \delta_{\lambda,\mu} n_\lambda c_\lambda$$

for some $n_\lambda \in \mathbb{C}\setminus\{0\}$.

Proof. □

5.6. **Theorem.** *Given a partition* λ,

(a) $c_\lambda^2 = n_\lambda c_\lambda$, *that is,* c_λ *is a scalar multiple of an idempotent.*
(b) $\mathbb{C}\mathfrak{S}_d \cdot c_\lambda$ *is an irreducible representation of* $\mathfrak{S}_d$, *say* V_λ.
(c) *Every irreducible representation of* $\mathfrak{S}_d$ *can be obtained in this way.*
(d) *Since conjugacy classes in* $\mathfrak{S}_d$ *are given by cycle type, which is encoded in a partition, this sets up a one-to-one correspondence between conjugacy classes of* $\mathfrak{S}_d$ *and irreducible representations of* $\mathfrak{S}_d$.

6. Two Sides of the Same Coin

Using symmetric function theory, we prove some results about the characters on $\mathfrak{S}_n$.

6.1. **Theorem** (Branching Rule)**.** *Let* $\mu \vdash n$. *Then,*

$$\operatorname{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \chi_\mu = \sum_{\lambda=\mu+an\ addable\ cell} \chi_\lambda$$

Similarly, $\lambda \vdash n$. *Then,*

$$\operatorname{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \chi_\lambda = \sum_{\mu=\lambda-a\ removable\ cell} \chi_\mu$$

Proof. The first statement follows from the Pieri rule. Namely,

$$\operatorname{ch}(\operatorname{Ind}_{\mathfrak{S}_{n-1}\times 1}^{\mathfrak{S}_n}(\chi_\mu \times \chi_{(1)})) = \operatorname{ch}(\chi_\mu)\operatorname{ch}(\chi_{(1)}) = s_\mu s_1 = h_1 s_\mu = \sum_{\lambda=\mu+\text{horizontal 1-strip}} s_\lambda$$

Thus giving us the result after taking ch^{-1}. The second result follows from Frobenius reciprocity. Namely,

$$\langle \operatorname{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \chi_\mu, \chi_\lambda\rangle = \langle \chi_\mu, \operatorname{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \chi_\lambda\rangle$$

□

6.2. **Theorem** (Young's Rule)**.** *If* $\lambda \vdash n$, *then the multiplicity of* S^μ *in* H_λ *is equal to* $K_{\mu\lambda}$

Proof. We know

$$h_\lambda = \sum_\mu K_{\mu\lambda} s_\mu \Longrightarrow K_{\mu,\lambda} = \langle h_\lambda, s_\mu\rangle = (H_\lambda, S^\mu)$$

□

6.3. **Theorem** (Murnaghan-Nakayama Rule)**.** *Given partitions $\lambda, \mu \vdash n$, the irreducible character χ_λ of $\mathfrak{S}_n$ has value on the conjugacy class of cycle type μ,*

$$\chi_\lambda(\mu) = \sum_{\mathsf{T}} (-1)^{\mathrm{ht}(\mathsf{T})}$$

where the sum is over all multi-ribbon tableaux with shape λ and weight μ.

Proof. If we take

$$\sum_{\lambda \vdash n} \chi_\lambda(\mu) s_\lambda = p_{\mu_1} \cdots p_{\mu_\ell} = \sum_{\mathsf{T} \text{ of weight } \mu} (-1)^{\mathrm{ht}(\mathsf{T})} s_{\mathrm{sh}(\mathsf{T})}$$

where T is a multi-ribbon tableau. (A ribon of length μ_1 labeled 1, a ribbon of length μ_2 labeled 2, and so on). Since the Schurs are a basis, gives

$$\chi_\lambda(\mu) = \sum_{\mathsf{T} \text{ of weight } \mu \text{ and shape } \lambda} (-1)^{\mathrm{ht}(\mathsf{T})}$$

□

References

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