SOME SPECIAL CLASSES OF PERMUTATIONS

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1. Introduction

Permutations are a fundamental object of study in combinatorics and are also studied in abstract algebra to define concrete examples of groups. In this monograph, we seek to explore some special classes of permutations that are interesting to combinatorialists. In particular, we will discuss

(a) 132-avoiding permutations,
(b) dominant partitions,
(c) Grassmannian partitions,
(d) and vexillary permutations.

Introduce some basic concepts like the Rothe diagram.

This monograph is currently very incomplete and simply a rough outline of details I would like to fully flush out at a later time. Currently, almost all results are stated as in [Man98].

2. 132-avoiding permutations

We first start with the general definition that

2.1. **Definition.** $\sigma \in \mathfrak{S}_n$ avoids pattern $\tau \in \mathfrak{S}_r$ (for $r \leq n$) if there is no subword $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_r}$ which "reduces" to τ.

2.2. **Example.** The permutation $\sigma = 24513$ (in one line notation) does not avoid 231 since $251 \mapsto 231$.

Thus, we immediately get the definition of a 132-avoiding permutation.

2.3. **Example.** In $\mathfrak{S}_3$, every permutation except 132 is 132-avoiding. In $\mathfrak{S}_4$, 1234 and 4123 are 132-avoiding.

Now, if we recall that the Catalan numbers C_n are given by either

$$C_n = \frac{1}{n+1}\binom{2n}{n} \quad \text{or} \quad \begin{cases} C_0 = 1 \\ C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \end{cases}$$

then we have

2.4. **Theorem.** *The number of* 132*-avoiding permutations in* $\mathfrak{S}_n$ *is given by* C_{n-1}.

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Proof. This is a classic combinatorics homework exercise and so will be left to the reader. Perhaps some hints will be given by the propositions that follow. $\square$

2.5. **Lemma.** *For a* 132*-avoiding permutation* σ*, there is no triple of integers* $i < j < k$ *such that* $\sigma_i < \sigma_k < \sigma_j$*.*

2.6. **Proposition.** *There is a bijection mapping* 132*-avoiding permutations in* $\mathfrak{S}_n$ *to Dyck paths of length* $2n$*.*

Proof. Given a Dyck path π with $(1,1)$ the top left square, we will construct a Rothe diagram for σ which is 132-avoiding.

(a) Place an X in each "removable corner" below the path π and put in the appropriate $\cdot$'s.
(b) Moving top to bottom, place X in westernmost empty sequence below π (and fill as appropriate). Note that since π never falls below the diagonal, there will be n rows and columns below π.

Note first that

$$\pi = UUUU\cdots UR\cdots RRRR \mapsto \begin{array}{|c|c|c|c|}\hline \times & \cdot & \cdot & \cdot \\ \hline \cdot & \times & \cdot & \cdot \\ \hline \cdot & \cdot & \times & \cdot \\ \hline \cdot & \cdot & \cdot & \times \\ \hline \end{array} \mapsto \sigma = id$$

which is 132-avoiding. Otherwise, if π is not of this form, then the cell $(1,1)$ is above π. Next, if $i < j < k$ and $\sigma_i < \sigma_j < \sigma_k$, we have a configuration with disconnected cells, which cannot be a Dyck path since no pair of cells can be disconnected.

$$\begin{array}{|c|c|c|c|c|c|}\hline & & & & \times & \cdot \\ \hline & & & & \cdot & \times \\ \hline \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & & \times & \cdot & \cdot & \cdot \\ \hline \cdot & \times & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \times & \cdot & \cdot \\ \hline \end{array}$$

$\square$

2.7. **Proposition.** *There is a bijection between* 132*-avoiding permutations and* 321*-avoiding permutations.*

3. Dominant Permutations

3.1. **Definition.** A permutations $\sigma \in \mathfrak{S}_n$ is a *dominant permutation* if it has partition Lehmer code, that is

$$c(\sigma) = (\ell_1, \ell_2, \dots, \ell_{n-1}) \text{ with } \ell_1 \geq \ell_2 \geq \cdots \geq \ell_{n-1}$$

3.2. **Example.** An example of a dominant permutation is $\sigma = 6324571 \in \mathfrak{S}_7$ since

$$c(\sigma) = (5, 2, 1, 1, 1, 1)$$

Interestingly, the Rothe diagram of σ given below exhibits the Ferrers diagram of the partition $5, 2, 1, 1, 1, 1$.

					$\times$	$\cdot$
		$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
	$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
	$\cdot$	$\cdot$	$\times$	$\cdot$	$\cdot$	$\cdot$
	$\cdot$	$\cdot$	$\cdot$	$\times$	$\cdot$	$\cdot$
	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\times$
$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

3.3. **Proposition.** *A permutation is dominant if and only if the cells above its Rothe diagram are the Ferrers diagram of a partition (in English convention).*

Proof. A common fact about Rothe diagrams is that the number of empty boxes in column i is the ith entry of the Lehmer code. Thus, since the transpose of the Rothe diagram is the Rothe diagram of σ^{-1} and such a transposition conjugates the embedded Ferrers diagram, we get that such a Rothe diagram gives a dominant permutation. Prove the converse.

$\square$

4. Grassmannian Permutations

4.1. **Definition.** A permutation $\sigma \in \mathfrak{S}_n$ is called *Grassmannian* if σ has at most one descent. In other words, there is at most one integer r such that

$$\sigma_1 < \sigma_2 < \cdots < \sigma_r \text{ and } \sigma_{r+1} < \sigma_{r+2} < \cdots < \sigma_n$$

4.2. **Example.** The permutation $\sigma = 245813679$ (in one line notation) has a descent at 8. Furthermore, it has code

$$c(\sigma) = (1, 2, 2, 5, 0, 0, 0, 0)$$

4.3. **Lemma.** *A Grassmannian permutation σ with Lehmer code $c(\sigma) = (\ell_1, \ldots, \ell_{n-1})$ has*

$$\ell_1 \leq \ell_2 \leq \cdots \leq \ell_r \leq n - r \text{ and } \ell_{r+1} = \cdots = \ell_{n-1} = 0$$

Proof. If either of these conditions fails, there must be more than one descent since a Grassmannian permutation must have entries (in one line notation) increasing up to σ_r and then increasing again from σ_{r+1} to σ_n. $\square$

4.4. **Proposition.** *Grassmannian permutations are in bijection with partitions inside an $r \times n - r$ rectangle.*

4.5. **Definition.** A permutation is called *bigrassmannian* if it is Grassmannian and its inverse is Grassmannian.

5. Vexillary Permutations

5.1. **Definition.** A permutations $\sigma \in \mathfrak{S}_n$ is called *vexillary* if it is 2143-avoiding. In other words, σ is vexillary if and only if there does not exist a sequence $i < j < k < l$ such that $\sigma_j < \sigma_i < \sigma_l < \sigma_k$.

5.2. **Lemma.** *A permutation σ is vexillary if and only if, up to a permutation of its rows and columns, its Rothe diagram has a diagram of a partition above it.*

Prove this

5.3. **Corollary.** *Dominant permutations and bigrassmannian permutations are vexillary.*

5.4. **Proposition.** *A permutation is vexillary if and only if $\lambda(\sigma) = \lambda(\sigma^{-1})'$, that is, the shape of σ is equal to the conjugate of the shape of σ^{-1}.*

5.5. **Definition.** The *flag* of a vexillary permutation σ is defined as follows.

(a) If $c(\sigma)_i \neq 0$, let e_i be the greatest integer $j \geq i$ such that $c(\sigma)_j \geq c(\sigma)_i$.

(b) The flag $\phi(\sigma)$ is the sequence of integers e_i ordered to be increasing.

5.6. **Example.** Let $\sigma = 126354$. Then, we have Rothe diagram

$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$				$\times$
$\cdot$	$\cdot$	$\times$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$		$\times$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\times$	$\cdot$	$\cdot$

We have $c(\sigma) = (0, 0, 3, 0, 1)$. Thus, $e_3 = 3$ and $e_5 = 5$ and so $\phi(\sigma) = (3, 5)$.

5.7. **Proposition.** *A vexillary permutation is completely determined by its shape and its flag.*

References

[Ful97] W. Fulton, *Young Tableaux*, 1997.

[Mac79] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 1979. 2nd Edition, 1995.

[Man98] L. Manivel, *Symmetric Functions, Schubert Polynomials, and Degeneracy Loci*, 1998. Translated by John R. Swallow; 2001.