# Quantum Groups
# Notes from a class taught by Weiqiang Wang in Fall 2019

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## 1. $q$-Numbers

Let $q$ be an indeterminate. Then, we will work in any of the following rings

$$\mathbb{Z}[q] \subseteq \mathbb{Z}[q, q^{-1}] \subseteq \mathbb{Q}(q) \subseteq \mathbb{C}(q)$$

1.1. Definition. For an indeterminate $q$ and $n \in \mathbb{Z}$, we define

(a) $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{1-n}$
(b) $[0]_q! := 1$ and $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$ for $n \in \mathbb{Z}_{\geq 0}$
(c) If $m \in \mathbb{Z}, n \geq 0$, then

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q[m-1]_q \cdots [m-n+1]_q}{[n]_q!} \overset{\text{If } m \geq 0}{=} \frac{[m]_q!}{[n]_q![m-n]_q!}$$

1.2. Remark. When the $q$ is clear, we will drop the $q$ from the notation and say $[n] := [n]_q$, etc.

1.3. Example. We compute some examples of $q$-numbers.

(a) $[0] = 0$
(b) $[1] = 1$
(c) $[2] = q + q^{-1}$

1.4. Proposition. *We have the following simple identities on q-numbers.*

*(a)* $[-n] = -[n]$ *for any* $n \in \mathbb{Z}$.
*(b)* $\begin{bmatrix} m \\ 0 \end{bmatrix}_q = 1 = \begin{bmatrix} m \\ m \end{bmatrix}_q$ *for all* $m \in \mathbb{Z}$.
*(c)* $\begin{bmatrix} m \\ n \end{bmatrix}_q = 0$ *for* $0 \leq m < n$.

1.5. Proposition. *We have the identity*

$$\begin{bmatrix} m+1 \\ n \end{bmatrix}_q = q^{-n} \begin{bmatrix} m \\ n \end{bmatrix}_q + q^{m-n+1} \begin{bmatrix} m \\ n-1 \end{bmatrix}_q$$

*and also that both* $[n]_q$ *and* $\begin{bmatrix} m \\ n \end{bmatrix}_q$ *are elements of* $\mathbb{Z}[q, q^{-1}]$

Proof. We compute directly that

$$\begin{aligned} q^{-n} \begin{bmatrix} m \\ n \end{bmatrix}_q + q^{m-n+1} \begin{bmatrix} m \\ n-1 \end{bmatrix}_q &= q^{-n} \frac{[m][m-1] \cdots [m-n+1]}{[n]_q!} + q^{m-n+1} \frac{[m][m-1] \cdots [m-n+2]}{[n-1]_q!} \\ &= q^{-n} \frac{[m][m-1] \cdots [m-n+1]}{[n]_q!} + q^{m-n+1} \frac{[n][m][m-1] \cdots [m-n+2]}{[n][n-1]_q!} \\ &= (q^{-n}[m-n+1] + q^{m-n+1}[n]) \frac{[m][m-1] \cdots [m-n+2]}{[n]_q!} \\ &= \left( \frac{q^{m-2n+1} - q^{-m-1}}{q - q^{-1}} + \frac{q^{m+1} - q^{m-2n+1}}{q - q^{-1}} \right) \frac{[m] \cdots [m-n+2]}{[n]_q!} \end{aligned}$$

$$= \left(\frac{q^{m+1} - q^{-(m+1)}}{q - q^{-1}}\right) \frac{[m] \cdots [m-n+2]}{[n]_q!}$$
$$= \frac{[m+1][m] \cdots [(m+1)-n+1]}{[n]_q!} = \begin{bmatrix} m+1 \\ n \end{bmatrix}_q$$

Now, observe that

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{q}{q^n} \cdot \frac{q^{2n} - 1}{q^2 - 1} = \frac{1}{q^{n-1}} (q^{2n-2} + q^{2n-4} + \cdots + q^2 + 1) \in \mathbb{Z}[q, q^{-1}]$$

This immediately gives that $[n]_q! \in \mathbb{Z}[q, q^{-1}]$. To show $\begin{bmatrix} m \\ n \end{bmatrix}_q \in \mathbb{Z}[q, q^{-1}]$, we proceed by induction on $m$. Namely, $\begin{bmatrix} m \\ 0 \end{bmatrix}_q = 1 \in \mathbb{Z}[q, q^{-1}]$ for all $m \in \mathbb{Z}$. Then,

$$\begin{bmatrix} m+1 \\ n \end{bmatrix}_q = q^n \underbrace{\begin{bmatrix} m \\ n \end{bmatrix}_q}_{\in \mathbb{Z}[q,q^{-1}]} + q^{-m+n-1} \underbrace{\begin{bmatrix} m \\ n-1 \end{bmatrix}_q}_{\in \mathbb{Z}[q,q^{-1}]} \in \mathbb{Z}[q, q^{-1}]$$

□

1.6. Theorem. *$q$-Binomial Theorem For an indeterminate $z$ and $r \geq 0$,*

$$\prod_{j=0}^{r-1} (1 + q^{2j} z) = \sum_{i=0}^{r-1} q^{i(r-1)} \begin{bmatrix} r \\ i \end{bmatrix}_q z^i$$

Proof. This follows by induction. If $r = 0$, then we simply have $1 = 1$. Now, proceed by induction. Then,

$$\prod_{j=0}^{r} (1 + q^{2j} z) = (1 + q^{2r} z) \left( \sum_{i=0}^{r-1} q^{i(r-1)} \begin{bmatrix} r \\ i \end{bmatrix}_q z^i \right) = \sum_{i=0}^{r-1} q^{i(r-1)} \begin{bmatrix} r \\ i \end{bmatrix}_q z^i + \sum_{i=0}^{r-1} q^{i(r-1)+2r} \begin{bmatrix} r \\ i \end{bmatrix}_q z^{i+1}$$

Then, if we fix the $z$ power for some $1 \leq k \leq r-1$, we get coefficient

$$q^{k(r-1)} \begin{bmatrix} r \\ k \end{bmatrix}_q + q^{(k-1)(r-1)+2r} \begin{bmatrix} r \\ k-1 \end{bmatrix}_q = q^{k(r-1)} \begin{bmatrix} r \\ k \end{bmatrix}_q + q^{k(r-1)+r+1} \begin{bmatrix} r \\ k-1 \end{bmatrix}_q$$
$$= q^{kr} \left( q^{-k} \begin{bmatrix} r \\ k \end{bmatrix}_q + q^{-k+r+1} \begin{bmatrix} r \\ k-1 \end{bmatrix}_q \right)$$
$$= q^{kr} \begin{bmatrix} r+1 \\ k \end{bmatrix}_q$$

where the last equality follows from 1.5. □

1.7. Corollary. *As consequences to 1.6, we get*

(a) *For $r \geq 1$,*

$$\sum_{i=0}^{r} (-1)^i q^{-i(r-1)} \left[ \begin{array}{c} r \\ i \end{array} \right]_q = 0$$

(b) *Assume $xy = q^2 yx$. Then,*

$$(x+y)^n = \sum_{t=0}^{n} q^{t(n-t)} \left[ \begin{array}{c} n \\ t \end{array} \right]_q y^t x^{n-t}$$

Sometimes in the literature, $q$-numbers are encoded slightly differently. We present the alternate definition here.

1.8. Definition. $\{n\}_v := 1 + v + v^2 + \cdots + v^{n-1} = \frac{v^n - 1}{v-1}$

Then, the two definitions are related as follows.

1.9. Proposition. *Setting $v = q^2$,*

$$\{n\}_v = q^{n-1}[n]_q$$

## 2. The Quantum Group $\mathcal{U}_q(\mathfrak{sl}_2)$

Throughout this section, we will let $\mathcal{U} := \mathcal{U}_q(\mathfrak{sl}_2)$. Let $\mathbb{k}$ be a field of characteristic 0 with $q \in \mathbb{k}$, $q \neq 0$, and $q$ is not a root of 1.

2.1. Definition. We define the *quantum group* $\mathcal{U} := \mathcal{U}_q(\mathfrak{sl}_2)$ to be the $\mathbb{k}$-algebra generated by elements $E, F, K, K^{-1}$ with relations

(a) $KK^{-1} = 1 = K^{-1}K$
(b) $KE = q^2 EK$
(c) $KF = q^{-1} FK$
(d) $EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$

2.2. Definition. We define the *Drinfield double* $\tilde{\mathcal{U}} = \langle E, F, K, K' \rangle$ to be the $\mathbb{k}$-algebra with relations

(a) $K'E = q^{-2} EK'$
(b) $K'F = q^2 EK'$
(c) $EF - FE = \frac{K - K'}{q - q^{-1}}$

2.3. Remark. Note that $\tilde{U}/\langle KK' - 1 \rangle \cong \mathcal{U}$ and that $KK'$ is central in $\tilde{U}$.

2.4. Definition. We define the following maps.

(a) The $\mathbb{k}$-linear involution $\omega$ acts on $\mathcal{U}$ by

$$\omega(E) = F, \omega(F) = E, \omega(K) = K^{-1}$$

(b) The $\mathbb{k}$-linear anti-involution $\tau$ (ie $\tau(xy) = \tau(y)\tau(x)$) acts on $\mathcal{U}$ by

$$\tau(E) = E, \tau(F) = F, \tau(K) = K^{-1}$$

2.5. Definition. For making computations more compact, we define the short hand

(a) $[K;n] = \frac{q^nK - q^{-n}K^{-1}}{q-q^{-1}}$
(b) For $n \in \mathbb{Z}_{\geq 0}$, $E^{(n)} = \frac{E^n}{[n]_q!}$ and $F^{(n)} = \frac{F^n}{[n]_q!}$.

2.6. Theorem (PBW Theorem). *$\{F^sK^nE^r \mid s, r \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}\}$ forms a basis for $\mathcal{U}$.*

Sketch of Proof. (a) Use the commutation relations of $\mathcal{U}$ to show that this is a spanning set; when commuting an $E$ past an $F$, one only picks up lower degree correction terms.
(b) Construct a "regular representation" $M = \Bbbk[\tilde{F}, \tilde{E}, \tilde{K}, \tilde{K^{-1}}]$ on which $\mathcal{U}$ acts to show linear independence. This argument is more sophisticated, but since this is a faithful representation, you get that the map $\theta\colon \mathcal{U} \to \mathrm{End}_{\Bbbk}(M)$ is injective and since $\theta(F^sK^nE^r)(1) = \tilde{F}^s\tilde{K}^n\tilde{E}^r$, which is known to be linearly independent, then the set $\{\theta(F^sK^NE^r)\}$ is linearly independent, thus giving us the desired linear independence by the injectivity of $\theta$. See [**Jan95**, Theorem 1.5].

$\square$

2.7. Lemma (Useful Identities). *(a)* $[K;n]E = E[K;n+2]$
*(b)* $[K;n]F = F[K;n-2]$
*(c)* $EF^s = F^sE + [s]F^{s-1}[K;1-s]$ *for* $s \geq 0$
*(d)* $E^rF^s = \sum_{i=0}^{\min(r,s)} \begin{bmatrix} r \\ i \end{bmatrix}_q \begin{bmatrix} s \\ i \end{bmatrix}_q [i]!F^{s-i}\left(\prod_{j=1}^{i}[K; i-(r+s)+j]\right)E^{r-i}$
*d'* $E^{(r)}F^{(s)} = \sum_{i=0}^{\min(r,s)} F^{(s-i)} \begin{bmatrix} K; 2i-(r+s) \\ i \end{bmatrix}_q E^{(r-i)}$ *where* $\begin{bmatrix} K; c \\ i \end{bmatrix}_q := \frac{[K;c][K;c-1]\cdots[K;c-i+1]}{[i]!}$.

Identity (d') gives one reason why divided powers are sometimes more convenient; writing identities with them can sometimes be simpler.

2.8. Remark. $\mathcal{U}_q(\mathfrak{sl}_2)$ has no zero-divisors.

### 2.1. Finite-dimensional Representations of $\mathcal{U}_q(\mathfrak{sl}_2)$.

2.9. Example. Let $M = \Bbbk m_0 \oplus \Bbbk m_1$ with $Km_0 = qm_0$ and $Km_1 = q^{-1}m_1$ and $E, F$ actions given by

$$\begin{tikzcd} 0 & m_1 \arrow[l, "F"'] \arrow[r, bend left, "E"] & m_0 \arrow[l, bend left, "F"] \arrow[r, "E"] & 0 \end{tikzcd}$$

2.10. Lemma. *Let $M$ be a finite-dimensional $\mathcal{U}$-module. Then, there exists an $r > 0$ such that $E^rM = 0$ and $F^rM = 0$.*

2.11. Definition. For $M \in \mathcal{U}$**-mod**, $\lambda \in \Bbbk^\times$, let $M_\lambda := \{m \in M \mid Km = \lambda m\}$ be called the $\lambda$-*weight subspace* of $M$.

2.12. Lemma. *(a)* $EM_\lambda \subseteq M_{q^2\lambda}$ *and* $FM_\lambda \subseteq M_{q^{-2}\lambda}$.
*(b) If* $M_\lambda \neq 0$ *and* $M$ *is simple, then*

$$M = \bigoplus_{n\in\mathbb{Z}} M_{q^{2n}\lambda}$$

2.13. Proposition. *Let* $M$ *be a finite-dimensional* $\mathcal{U}$*-module. Then,*

$$M = \bigoplus_{a\in\mathbb{Z}} M_{+q^a} \oplus M_{-q^a}$$

Proof. It is equivalent to show that the minimal polynomial of $K$ on $M$ is of the form $\prod_i (K - \lambda_i)$ with distinct $\lambda_i \in \pm q^{\mathbb{Z}}$. To do this, set

$$h_r := \prod_{j=1-r}^{r-1} [K; r-s+j], \quad r > 0, h_0 = 1$$

Now, for $s > 0$, if $F^s M = 0$, then $F^{s-r} h_r M = 0$ for all $0 \leq r \leq s$ because

$$\left(E^r F^s \prod_{j=1}^{r-1} [K; r-s+j]\right) M = \left(\sum_{i=0}^{r} a_i F^{s-i} h_i \prod_{j=0}^{r-i-1} [K; -s-j] E^{r-i}\right) M$$

for $a_i = \begin{bmatrix} r \\ i \end{bmatrix}_q \begin{bmatrix} s \\ i \end{bmatrix}_q [i]!$ by 2.7(d) allows us to use induction. Then, we have

$$0 = h_s M = \prod_{j=1-s}^{s-1} \left[ \underbrace{(q-q^{-1})^{-1} q^j K^{-1}}_{\text{Invertible scalar}} \quad \underbrace{(K^2 - q^{-2j})}_{\text{Minimal polynomial divides this}} \right] M$$

and thus we have distinct $\lambda_i \in \pm q^{\mathbb{Z}}$ $\square$

# Bibliography

[Jan95] J. C. Jantzen, *Lectures on Quantum Groups*, 1995.