Springer Theory: Lecture notes from a reading course

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1. Introduction (presented by Weiqiang Wang)

Yarrrrr pirates!

(8/28/2017) Lecture 1: Semisimple Lie Algebras and Flag Varieties (presented by Mike Reeks).

2. Semisimple Lie Algebras and Flag Varieties

In this section, we will review essential facts for our study going forward. Fix G to be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$.

2.1. Proposition. *$\mathfrak{g}$ is a G-module via the adjoint action. That is, for $g \in G$, $x \in \mathfrak{g}$*

$$g.x = \mathrm{Ad}_d(\phi_g)_I(x)$$

where $\phi_g \colon G \to G$ maps $h \mapsto ghg^{-1}$ for $h \in G$.

Proof. $\square$ Add in a proof here

2.2. Definition. A maximal solvable subgroup of G is called a *Borel* subgroup.

2.3. Definition. A torus of a compact Lie group G is a compact, connected, abelian Lie subgroup of G.

2.4. Proposition. *Given a Borel subgroup $B \leq G$ of a Lie group, B has a maximal torus $T \leq G$ and a unipotent radical $U \leq G$ such that $T \cdot U = B$.*

Proof. $\square$ Add in a proof here

2.5. Example. Let $G = SL_n(\mathbb{C})$. Then,

$$B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}, \ T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}, \ U = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

In particular, $B = TU$.

2.6. Definition. When considering the tangent spaces at the identity, we get a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ where $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{n}$ is a nilradical.

2.7. Remark. Note that $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ and $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \cong \mathfrak{h}$.

2.8. Definition. We define the *rank of a Lie algebra* $\mathfrak{g}$ to be

$$\operatorname{rank} \mathfrak{g} := \dim \mathfrak{h}$$

2.9. Lemma. *(a) Let $B \leq G$ be Borel. Then, $B = N_G(B)$ (that is, B is self-normalizing).*
(b) All Borel subgroups are G-conjugate

Proof. $\square$ Add a proof here

2.10. Definition. For $x \in \mathfrak{g}$, we define

$$Z_{\mathfrak{g}}(x) = \{y \in \mathfrak{g} \mid [x, y] = 0\}, \ Z_G(x) = \{g \in G \mid g.x = x\}$$

2.11. Example. Let $\mathfrak{g} \subseteq \mathfrak{gl}_2(\mathbb{C})$ be all 2 by 2 upper-triangular matrices. Then,

2.12. Proposition. $\dim(Z_{\mathfrak{g}}(x)) \geq \operatorname{rank} \mathfrak{g}$

Proof. Let x be semisimple. Then, x is contained in some Cartan subalgebra $\mathfrak{h}$. However, $\mathfrak{h} \subseteq Z_{\mathfrak{g}}(x)$. Thus, the proposition is true for x semisimple. However, semisimple elements are Zariski dense, so the result applies for the closure. $\square$

Check this last thing.

2.13. Definition. We say $x \in \mathfrak{g}$ is *regular* if $\dim Z_{\mathfrak{g}}(x) = \operatorname{rank} \mathfrak{g}$.

2.14. Definition. We say $x \in \mathfrak{g}$ is *semisimple (resp. nilpotent)* if $\operatorname{ad}(x)$ is semisimple (resp nilpotent).

2.15. Remark. This result is actually somewhat non-trivial, but is covered by Engel's theorem and related results.

Now, recall that any $x \in \mathfrak{g}$ has a Jordan decomposition $x = s + n$ where $s \in \mathfrak{g}$ is semisimple and $n \in \mathfrak{g}$ is nilpotent and s, n commute.

2.16. Definition. Let $\mathfrak{g}^{sr}$ be the set of semisimple regular elements in $\mathfrak{g}$.

2.17. Example. Note that not all regular elements are semisimple. Take

$$x = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

Such an element is regular in $\mathfrak{sl}_n$ because $Z_{\mathfrak{g}}(x)$ is spanned by the basis $\{x, x^2, \ldots, x^{n-1}\}$, which is the same size as the basis $\{e_{ii} - e_{i+1,i+1}\}_{i=1}^{n-1}$ that spans the standard Cartan subalgebra$\mathfrak{h}$ in $\mathfrak{sl}_n$. However, since it is nilpotent, it cannot be semisimple.

2.18. Proposition. *Fix $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$. Then,*

- *(a) Any element of $\mathfrak{h}$ is semisimple and any semisimple element is G-conjugate to an element of $\mathfrak{h}$.*
- *(b) If $x \in \mathfrak{g}^{sr}$, then $Z_{\mathfrak{g}}(x)$ is a Cartan subalgebra.*
- *(c) $\mathfrak{g}^{sr}$ is a G-stable dense subset of $\mathfrak{g}$ (in the Zariski topology)*

"Proof". Any element of $\mathfrak{h}$ is semisimple by definition. Also, in Humphreys chapter 16, he gives an argument that all Cartan subalgebras are G-conjugate using the conjugacy of Borel subalgebras.

For the second claim, $Z_{\mathfrak{g}}(x)$ is straightforwardly a subalgebra. Varadarajan gives an argument for why it is a Cartan subalgebra. $\square$ reproduce this here.

2.19. Example. A typical example of part (b) in $\mathfrak{sl}_n$ is given by

$$x = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

with λ_i distinct. Then, $Z_{\mathfrak{g}}(x)$ is given by matrices (a_{ij}) such that

$$0 = [(a_{ij}), x] = \sum_{i,j} [a_{ij}e_{ij}, x] = \sum_{ij} (a_{ij}\lambda_j e_{ij} - \lambda_i a_{ij} e_{ij})$$

So, if $i \neq j$, then $a_{ij} = 0$ since $\lambda_i \neq \lambda_j$. That is to say, $Z_{\mathfrak{g}}(x)$ is all diagonal matrices in $\mathfrak{sl}_n$

2.20. Lemma. *There exists a G-invariant polynomial P with coefficients in $\mathfrak{g}$ such that*

$$x \in \mathfrak{g}^{sr} \iff P(x) \neq 0$$

Proof. Take $x \in \mathfrak{g}$. Since $\dim \ker(\mathrm{ad}_x) \geq \operatorname{rank} \mathfrak{g}$, the characteristic polynomial of ad_x is given by

$$\det(tI - \mathrm{ad}_x) = t^r P_r(x) + t^{r+1} P_{r+1}(x) + \cdots$$

where P_i are G-invariant polynomials on $\mathfrak{g}$ and $r = \operatorname{rank} \mathfrak{g}$. So, we have that x is regular if and only if $t = 0$ is a zero of order r of the characteristic polynomial. However, this is possible if and only if $P_r(x) \neq 0$. Thus, for x is semisimple and regular, we have found a desired polynomial. Since this polynomial is G-invariant and all semisimple elements are conjugate to an element in $\mathfrak{h}$, $\square$ Figure out why this polynomial applies to all regular semisimple elements.

Flag Variety.

2.21. Definition. Define $\mathcal{B}$ to be the set of all Borel subalgebras of $\mathfrak{g}$.

2.22. Proposition. *$\mathcal{B}$ is a projective variety.*

2.23. Definition. Let Gr be the Grassmanian of $\dim \mathfrak{b}$-dimensional subspaces of $\mathfrak{g}$.

2.24. Proposition. *$\mathcal{B} \subseteq \mathrm{Gr}$ as a closed subvariety of solvable subalgebras.*

2.25. Proposition. *The stabilizer of $\mathfrak{b}$ under the G-adjoint action is B, the Borel subgroup of G.*

2.26. Proposition. *For $g \in G$, the map $g \mapsto g \cdot \mathfrak{b} \cdot g^{-1}$ induces a bijection*

$$G/B \to \mathcal{B}$$

Moreover, this is a G-equivariant isomorphism of varieties.

2.27. Example. In type A, $\mathcal{B} = \{F = (0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n = \mathbb{C}^n)\}$ where $\dim F_i = i$, the variety of complete flags.

2.28. Proposition. *Fix $B \leq G$ with Lie algebra $\mathfrak{b}_0$. Then, we have the following bijections.*

$$N_G(T)/T = W_T \xrightarrow{3} BnG/B \xrightarrow{1} \{B\text{-orbits on } \mathcal{B}\} \xrightarrow{2} \{G\text{-diagonal orbits on } \mathcal{B} \times \mathcal{B}\}$$

where

- *1 is given by $B \cdot g \cdot B \mapsto B(g \cdot B/B)$,*
- *2 is given by $B.\mathfrak{b} \mapsto G.(\mathfrak{b}_0, \mathfrak{b})$, and*
- *3 is given by $W \mapsto B \cdot \overline{w} \cdot B$ where $\overline{w}$ is a representative of W in $N_G(T)$.*

Get some handle on the proofs of these propositions!

2.29. Corollary. *As a result, $\mathcal{B} = \bigsqcup_{w \in W} \mathcal{B}_w$, where $\mathcal{B}_w$ is a Bruhat cell.*

Understand this better. Why does this follow and why do we care?

So W is a fixed point set of the $\mathbb{C}^*$ action on $\mathcal{B}$. $w \in W$ is $w \cdot B/B$, $w \in W_T$. So, $\mathcal{B}_W = B \cdot (wB/B)$ gives us

$$G/B = \mathcal{B} = \bigsqcup_{w \in W} \mathcal{B}_W = \bigsqcup BwB/B$$

2.30. Corollary. *Each B-orbit in $\mathcal{B}$ contains exactly one point of the form wB/B, $w \in W_T$.*

Make sense of this statement.

2.31. Proposition (Plucker embedding). *Skipped for now.*

2.0.1. *Extended $\mathfrak{sl}_n$ example.* Let $G = SL_n$ and $\mathfrak{g} = \mathfrak{sl}_n$. Then, we have the following result.

2.32. Lemma. *$\mathcal{B}$ is naturally identified with the variety of complete flags.*

Proof. Let F be a complete flag. Then,

$$\mathfrak{b}_F = \{x \in \mathfrak{sl}_n \mid x(F_i) \subseteq F_i, \forall i\}$$

For example, if F is the coordinate flag, that is $F = 0 \subsetneq \mathbb{C}^1 \subsetneq C^2 \subsetneq \cdots \subsetneq \mathbb{C}^n$, then

$$\mathfrak{b}_F = \mathfrak{b} = \begin{pmatrix} & & * \\ & \ddots & \\ & & \end{pmatrix}$$

Note that any 2 flags are conjugate by the SL_n action, so $\mathfrak{b}_F$ is Borel for every flag F. Thus, our map is surjective. Hence, it is bijective and an isomorphism of algebraic varieties □

why does this last statement follow?

2.33. LEMMA. *Consider $\mathbb{C}^n/S_n$. It is isomorphic as a variety to $\mathbb{C}[\lambda]_{n-1}$, polynomials in λ of degree less than or equal to $n-1$.*

PROOF. Consider the map $\psi\colon \mathbb{C}^n \to \mathbb{C}[\lambda]_{n-1}$ given by

$$(x_1, \dots, x_n) \mapsto \lambda^n - \prod(\lambda - x_i)$$

The result of this map does not depend on the order of the x_i's, so we can mod out by the action of S^n on this map. $\square$

Now, given a linear map $x\colon \mathbb{C}^n \to \mathbb{C}^n$, we have an unordered n-tuple of eigenvalues $\{x_i\}$, but if $x \in \mathfrak{sl}_n$, we know that $\sum x_i = 0$ since $\operatorname{tr} x = 0$. So, take

$$\mathbb{C}^{n-1} \cong \{(x_1, \dots, x_n) \mid \sum x_i = 0\}$$

as the $n-1$-dimensional hyperplane in $\mathbb{C}^n$. It is still stable under the S^n-action, giving us a map $\phi\colon \mathfrak{sl}_n \to \mathbb{C}^{n-1}/S_n$ given by

$$x \mapsto (x_1, \dots, x_n), \text{ the eigenvalues of } x$$

2.34. DEFINITION. Given the information above, for $\mathfrak{g} = \mathfrak{sl}_n$, we define the *incidence variety of* $\mathfrak{g}$, denoted $\tilde{\mathfrak{g}}$, to be

$$\tilde{\mathfrak{g}} := \{(x, F) \in \mathfrak{sl}_n \times \mathcal{B} \mid x(F_i) \subseteq F_i\}$$

2.35. PROPOSITION. *The following diagram commutes*

$$\begin{tikzcd}
& \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\
\mathfrak{g} \arrow[dr, "\phi"'] & & \mathbb{C}^{n-1} \subseteq \mathbb{C}^n \arrow[dl, "\psi"] \\
& \mathbb{C}^{n-1}/S_n &
\end{tikzcd}$$

where $\nu\colon \tilde{\mathfrak{g}} \to \mathbb{C}^{n-1}$ sends (x, F) to the ordered list of its eigenvalues λ_i where λ_i is the eigenvalue of the induced map $F_i/F_{i-1} \to F_i/F_{i-1}$ and $\mu\colon \tilde{\mathfrak{g}} \to \mathfrak{g}$ is the standard projection onto the first coordinate $(x, F) \mapsto x$.

2.36. DEFINITION. Let $\mathcal{B}_x := \{F \in \mathcal{B} \mid x(F_i) \subseteq F_i, \forall i\}$.

2.37. PROPOSITION. *For all $x \in \mathfrak{g}^{sr}$, the set $\mathcal{B}_x$ consists of $n-1$ points and has a canonical free S_n-action.*

PROOF. Look at the eigenspaces. Note that $\nu|_{\tilde{\mathfrak{g}}^{sr}} = \mu^{-1}(\mathfrak{g}^{sr})$ commutes with the S_n-action. $\square$

Actually do the proof.

2.38. THEOREM (Universal resolution for general $\mathfrak{g}$). *Let $\mathfrak{g}$ be an arbitrary Lie algebra. Then, we have incidence variety $\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid x \in$*

$\mathfrak{b}\}$ *such that the following diagram commutes*

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathfrak{g} \arrow[dr, "\rho"'] & & \mathfrak{h} \arrow[dl, "\pi"] \\ & \mathfrak{h}/W & \end{tikzcd}$$

where $\nu\colon \tilde{\mathfrak{g}} \to \mathfrak{h}$ *is given by* $(x, \mathfrak{b}) \mapsto x \mod [\mathfrak{b}, \mathfrak{b}]$ *(recall* $\mathfrak{h} = \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$*) and* $\rho\colon \mathfrak{g} \to \mathfrak{h}/W$ *induces embedding*

$$\mathbb{C}[\mathfrak{h}]^W \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$$

for Weyl group W*.*

(9/4/2017) Lecture 2 (presented by Chris Leonard). Recall that, given a torus $T \leq G$, we get a Cartan subalgebra that acts on itself by conjugation? This gives rise to a Weyl group $W_T = N_G(T)/T$.

2.39. Lemma. *If* $\mathfrak{b}, \mathfrak{b}' \in \mathcal{B}$*, then there is a canonical isomorphism*

$$\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \xrightarrow{\sim} \mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}']$$

which is independent of choice of conjugation.

2.40. Definition. We define $\mathfrak{H} = \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ to be the *abstract Cartan subalgebra.*

2.41. Proposition. *Normally, we take a root system on the pair* $(\mathfrak{h}, \mathfrak{b})$ *but this gives rise to a root system on* $\mathfrak{H}$ *and an abstract Weyl group* $\mathbb{W}$*. Thus, the* W *action on* $\mathfrak{h}$ *gives rise to a* $\mathbb{W}$ *action on* $\mathfrak{H}$*.*

2.42. Theorem (Universal resolution for general $\mathfrak{g}$ revisited). *Let* $\mathfrak{g}$ *be an arbitrary Lie algebra. Then, we have incidence variety* $\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}\}$ *such that the following diagram commutes*

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathfrak{g} \arrow[dr, "\rho"'] & & \mathfrak{H} \arrow[dl, "\pi"] \\ & \mathfrak{H}/\mathbb{W} & \end{tikzcd}$$

where $\nu\colon \tilde{\mathfrak{g}} \to \mathfrak{H}$ *is given by* $(x, \mathfrak{b}) \mapsto x \mod [\mathfrak{b}, \mathfrak{b}]$

2.43. Theorem (Chevalley's Restriction Theorem).

$$\mathbb{C}[\mathfrak{h}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^{W_T} = \mathbb{C}[\mathfrak{H}]^{\mathbb{W}}$$

for abstract Weyl group $\mathbb{W}$*.*

2.44. Corollary. *From the above, using the Nullstenllensatz (Proposition 2.2.2 in* [**CG00**]*), we get*

$$\mathfrak{g} = \operatorname{Specm} \mathbb{C}[\mathfrak{g}] \to \operatorname{Specm} \mathbb{C}[\mathfrak{H}]^{\mathbb{W}} = \mathfrak{H}/\mathbb{W}$$

where this composition is ρ in the Universal Resolution.

Make more sense of how this and the above two theorems fit together. Currently, this is a mess.

2.45. Definition. We say $\mathfrak{h} \in \mathfrak{H}$ is *regular* if $|\mathbb{W} \cdot h| = |\mathbb{W}|$. We define $\mathfrak{H}^{reg}$ to be the collection of regular elements of $\mathfrak{H}$. Finally, we define $\tilde{\mathfrak{g}}^{sr} = \mu^{-1}(\mathfrak{g}^{sr}) = \nu^{-1}(\mathfrak{H}^{reg})$, where μ, ν are from the Universal Resolution above.

2.46. Proposition. *(*[**CG00**] *Prop 3.1.36) $\mu\colon \tilde{\mathfrak{g}}^{sr} \to \mathfrak{g}^{sr}$ is a principal $\mathbb{W}$-bundle, that is $\mathbb{W}$ acts freely on $\mu^{-1}(x)$ and so $\tilde{\mathfrak{g}}^{sr}$ looks like $\mathfrak{g}^{sr} \times \mathbb{W}$.*

Proof. There exists a unique Cartan subalgebra $\mathfrak{h} = Z_{\mathfrak{g}}(x)$. We then recall that $\mathbb{W}$ acts freely and transitively on Borel subalgebras containing $\mathfrak{h}$, so $\mathbb{W}$ acts freely on $\mu^{-1}(x)$, the set of all Borel subalgebras containing x. $\square$

Why? Must all Borel subalgebras containing x also contain $\mathfrak{h}$?

3. The Nilpotent Cone

3.1. Definition. Let $\mathcal{N} = \{\text{nilpotent elements in } \mathfrak{g}\}$. We define the *nilpotent cone* to be

$$\tilde{\mathcal{N}} = \mu^{-1}(\mathcal{N}) = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\}$$

where $\mu\colon \tilde{g} \to \mathfrak{g}$ is the projection from 2.42.

3.2. Proposition. *The projection $\pi\colon \tilde{\mathcal{N}} \to \mathcal{B}$ makes $\tilde{\mathrm{N}}$ into a vector bundle over $\mathcal{B}$ with fiber $\pi^{-1}(\mathfrak{b}) = \mathfrak{n}$.*

3.3. Proposition. *Fix $B, \mathfrak{b}$. We can define a group action of B on $G \times \mathfrak{b}$ given by*

$$b \cdot (g, x) = (gb^{-1}, bxb^{-1})$$

3.4. Definition. Let $G \times_B \mathfrak{b}$ be the set of orbits of $G \times \mathfrak{b}$ under the B-action defined above.

3.5. Proposition. *$G \times_B \mathfrak{b}$ is a G-equivariant vector bundle whose fibres are $\mathfrak{b}$*

$$\begin{tikzcd} G \times_B \mathfrak{b} \arrow[d] & (g, x) \arrow[d, mapsto] \\ G/B & g.B/B \end{tikzcd}$$

3.6. Proposition. *The following diagrams commute*

$$\begin{tikzcd} G \times_B \mathfrak{b} \arrow[r, "\sim"] \arrow[d] & \tilde{g} \arrow[d] & G \times_B \mathfrak{n} \arrow[r, "\sim"] \arrow[d] & \tilde{\mathcal{N}} \\ G/B \arrow[r, "\sim"] & \mathcal{B} & G/B \arrow[r, "\sim"] & \mathcal{B} \end{tikzcd}$$

From this, we can conclude that $\tilde{\mathrm{N}}$ is a smooth variety.

Why?!

We now go into some necessary background for understanding our nilpotent cone.

3.1. Symplectic Manifolds.

3.7. Definition. A *symplectic manifold* M is a holomorphic manifold with a non-degenerate closed 2-form ω such that, for all $x \in M$,

$$\omega_x \colon T_xM \times T_xM \to \mathbb{C}$$

is symplectic.

3.8. Example. Consider $\mathbb{C}^{2n}$ with coordinates $p_1, \ldots, p_n, q_1, \ldots, q_n$ and $\omega = \sum dp_i \wedge dq_i$. Then, with correct choice of basis for $T_x\mathbb{C}^{2n}$, we get

$$\omega_x = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

3.9. Theorem. *Locally, all symplectic manifolds look like the example above.*

3.10. Definition. We define $\mathcal{O}(M)$ to be the set of all *regular functions* on M, that is, all 0-forms.

3.11. Definition. Let $V(M)$ be the set of all vector fields on M. That is, $\eta \in V(M)$ is given by $\eta \colon M \to TM$ with smooth choice $x \mapsto \eta_x \in T_xM$.

3.12. Definition. We define $\zeta \colon \mathcal{O}(M) \to V(M)$ by

$$\omega_x(-, (\zeta_f)_x) = df_x$$

What is this? $\zeta_f = \zeta(f)$? Also, where does the $-$ go?

3.13. Definition. Define $\{,\}$ on $\mathcal{O}(M)$ by

$$\{f, g\} = \omega(\zeta_f, \zeta_g)$$

3.14. Theorem. *$\mathcal{O}(M)$ is a Poisson algebra. That is, $\mathcal{O}(M)$ is an associative algebra, a Lie algebra, and $\{a, -\} \colon \mathcal{O}(M) \to \mathcal{O}(M)$ satisfies a Liebniz rule: $\{a, b \cdot c\} = \{a, b\} \cdot c + b \cdot \{a, c\}$.*

3.15. Example. Take $M = \mathbb{C}^2$ with coordinates p, q and $\omega = dp \wedge dq$. Then, we can represent $\omega_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Write $\eta \in V(M)$ as

$$\eta = \begin{pmatrix} \phi(p,q) \\ \psi(p,q) \end{pmatrix} : M \to T_{(p,q)}M = \mathbb{C}^2$$

Then, $\omega(-, \eta) = (\psi, -\phi) \colon M \times \mathbb{C}^2 \to \mathbb{C}$ and gives $df = (\frac{\partial f}{\partial p}, \frac{\partial f}{\partial q})$. We then note that

Why is the domain $M \times \mathbb{C}^2$?

$$\zeta_f = \begin{pmatrix} -\frac{\partial f}{\partial q} \\ \frac{\partial f}{\partial p} \end{pmatrix} \implies \zeta_{q^2/2} = \begin{pmatrix} -q \\ 0 \end{pmatrix}, \quad \zeta_{-p^2/2} = \begin{pmatrix} 0 \\ -p \end{pmatrix}, \quad \zeta_{pq} = \begin{pmatrix} -p \\ q \end{pmatrix}$$

Thus, we get

$$\begin{aligned}
\{\frac{q^2}{2}, -\frac{p^2}{2}\} &= \omega\left(\begin{pmatrix}-q\\0\end{pmatrix}, \begin{pmatrix}0\\-p\end{pmatrix}\right) = pq \\
\{pq, \frac{q^2}{2}\} &= q^2 = 2\frac{q^2}{2} \\
\{pq, -\frac{p^2}{2}\} &= p^2 = -2\frac{p^2}{2}
\end{aligned}$$

So, we have a Lie algebra embedding $\mathfrak{sl}_2 \hookrightarrow \mathcal{O}(M)$ given by

$$\begin{aligned}
e &\mapsto \frac{q^2}{2} \\
f &\mapsto -\frac{p^2}{2} \\
h &\mapsto pq
\end{aligned}$$

3.16. Definition. For $\eta \in V(M)$, define $L_\eta\colon \mathcal{O}(M) \to \mathcal{O}(M)$ by

$$(L_\eta f)(m) = df_m(y_m)$$

Say something about ζ mapping $\mathcal{O}(M) \to V(M)$ and getting a Lie algebra.

3.17. Proposition. *The map $V(M) \to \mathrm{Der}(\mathcal{O}(M))$ defined by $\eta \mapsto L_\eta$ is an isomorphism of Lie algebras where we define $[\eta, \eta'] \in V(M)$ and $L_{[\eta,\eta']} = [L_\eta, L_{\eta'}]$.*

Where did this y_m come from? What is it?

3.2. The Moment Map.

3.18. Proposition. *A Lie group G acts on symplectic manifold M via a symplectic action $\phi_m\colon G \to M$ given by $g \mapsto gm$. This induces $T_eG = \mathfrak{g}$ action on T_mM for all m given by Lie algebra homomorphism $\mathfrak{g} \to V(M)$.*

Write this out more explicitly.

3.19. Definition. A symplectic G-action is *Hamiltonian* if

$$\begin{tikzcd}
 & \mathfrak{g} \arrow[dl, dashed, "\exists H"'] \arrow[dr] & \\
\mathcal{O}(M) \arrow[rr, "\zeta"] & & V(M)
\end{tikzcd}$$

We denote $H(x) = H_x$.

3.20. Definition. We define *the moment map* $\mu\colon M \to \mathfrak{g}^*$ by

$$\begin{aligned}
\mu(m)\colon \mathfrak{g} &\to \mathbb{C} \\
x &\mapsto H_x(m)
\end{aligned}$$

3.21. Example. Take $M = \mathbb{C}^2$ and let $SP_2 = SL_2$ act on M. Then, we get Hamiltonian $H\colon \mathfrak{sl}_2 \to \mathcal{O}(M)$ by example 3.15. We also have a canonical isomorphism between $\mathfrak{sl}_2 \xrightarrow{\sim} \mathfrak{sl}_2^*$ given by $e \mapsto e^*, f \mapsto f^*, h \mapsto h^*$. Thus, we have moment map $\mu\colon M \to \mathfrak{g}$ given by

$$\mu(p,q) = \frac{1}{2}\begin{pmatrix} pq & -p^2 \\ q^2 & -pq \end{pmatrix} \in \mathcal{N} \text{ because } \det = 0$$

This is precisely equal to $\frac{q^2}{2}e^* - \frac{p^2}{2}f^* + pqh^*$, so $\mu\colon \mathbb{C}^2 \to \mathcal{N}$ is a 2-fold cover ramified at the origin.

ramified?!

3.22. Proposition. *If M is a manifold, then the cotangent bundle T^*M is symplectic.*

What does it mean for a cotangent bundle to be symplectic?

3.23. Proposition. *The action C_T on M induces an action of G on T^*M, which gives us our Hamiltonian.*

What is C_T? What are these actions?

3.24. Example. Take $P \leq G$. Then, G acts on G/P and so G acts on $T^*(G/P)$. Let $\mathfrak{p}^\perp \subseteq \mathfrak{g}^*$ where $\mathfrak{p}^\perp$ is the annihilator of $\mathfrak{p}$.

how?

3.25. Proposition. *([CG00] Prop 1.4.9/10) In the case of the above example, there is a natural G-equivariant isomorphism*

$$\begin{tikzcd} T^*(G/P) \arrow[rr, "\sim"] \arrow[dr] & & G \times_{\mathfrak{p}} \mathfrak{p}^\perp \arrow[dl] \\ & G/P & \end{tikzcd}$$

and the moment map is given by

$$\begin{aligned} \mu\colon G\times_{\mathfrak{p}} &\to \mathfrak{g}^* \\ (g,\alpha) &\mapsto g\alpha g^{-1} \end{aligned}$$

This is great, because it is explicit, but where does this come from?

3.3. Return to the Nilcone.

3.26. Lemma. *Recall that we can identify $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ via the Killing form.*

Recall this.

3.27. Proposition. *([CG00] Prop 3.22/3) There is a natural G-equivariant isomorphism*

$$\tilde{\mathcal{N}} \cong T^*\mathcal{B}$$

and $\mu\colon R^*\mathcal{B} \to \mathcal{N}$ is the moment map from G acting on $\mathcal{B}$ and it is surjective.

3.28. Definition. The moment map above is called *Springer's resolution.*

Proof of proposition.

Understand why these are actually isomorphisms.

$$\begin{aligned} T * \mathcal{B} &\cong T^*(G/B) \\ &\cong G \times_B \mathfrak{b}^\perp \\ &\cong G \times_B \mathfrak{n} \\ &\cong \tilde{\mathcal{N}} \end{aligned}$$

$\square$

3.29. Proposition. *([CG00] Prop 3.2.5) $x \in \mathfrak{g}$ is nilpotent if and only if $P(x) = 0$ for all $P \in \mathbb{C}[\mathfrak{g}]^G_+$ (which is polynomials in $\mathbb{C}[\mathfrak{g}]^G$ with no constant term).*

3.30. Definition. $x \in \mathfrak{g}^*$ is nilpotent if and only if $P(x) = 0$ for all $P \in \mathbb{C}[\mathfrak{g}^*]^G_+$.

Proof of proposition. Consider the universal resolution 2.42,

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl] \arrow[dr] & \\ \mathfrak{g} \arrow[dr, "\rho"] & & \mathfrak{H} \arrow[dl] \\ & \mathfrak{H}/\mathbb{W} & \end{tikzcd}$$

with $\rho^{-1}(0) = \mathcal{N}$. $\square$

Does this follow immediately or am I missing something obvious here?

3.31. Corollary. *([CG00] Cor 3.2.8) $\mathcal{N}$ is an irreducible variety of dimension $2 \dim \mathfrak{n}$.*

3.32. Proposition. *([CG00] Prop 3.2.10) Regular nilpotent elements for a single open Zariski-dense orbit in cN, denoted $\mathcal{O}_{prin}$, the principal orbit.*

3.33. Example. Let $\mathfrak{g} = \mathfrak{sl}_n$. Then, the orbit is the orbit of

$$\begin{pmatrix} 0 & 1 & \\ & \ddots & \ddots \\ & & 1 \\ & & 0 \end{pmatrix}$$

3.34. Proposition. *Any regular nllpotent element is contained in a unique Borel subalgebra.*

3.35. Remark. Given $\mu\colon \tilde{\mathcal{N}} \to \mathcal{N}$, μ is a resolution of singularities of $\mathcal{N}$, that is it restricts to an isommorphism on an open dense subset.

(9/11/2017) Lecture 3: The Steinberg Variety (presented by Chris Chung).

3.36. Corollary. *([CG00] Cor 3.2.25) $\mathcal{N} = \bigsqcup \mathcal{O}$ is an* algebraic stratification *of $\mathcal{N}$, each $\mathcal{O}$is a smooth locally closed subvariety of $\mathcal{N}$.*

4. The Steinberg Variety

4.1. Definition. We define the *Steinberg variety* to be

$$Z := \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} = \{(x, \mathfrak{b}, x', \mathfrak{b}') \in \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \mid x = x'\}$$

which is isomorphic to

$$\{(x, \mathfrak{b}, \mathfrak{b}') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathfrak{b} \cap \mathfrak{b}'\}$$

4.2. Proposition. *From the standard resolution, we get the diagram*

$$\begin{tikzcd} & Z \arrow[dl, "\mu_Z"'] \arrow[dr, "\pi^2"] & \\ \mathcal{N} & & \mathcal{B} \times \mathcal{B} \end{tikzcd}$$

and so we have $\tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \cong T^*\mathcal{B} \times T^*\mathcal{B} \overset{sign}{\cong} T^*(\mathcal{B} \times \mathcal{B})$*, that is, isomorphic up to a sign to account for the symplectic form* $\omega = p_1^*\omega_1 - p_2^*\omega_2$ *where* ω_1 *and* ω_2 *account for the standard symplectic forms on* $T^*\mathcal{B}$*.*

Thus, we can think of the following

$$\begin{tikzcd} & (T^*(\mathcal{B} \times \mathcal{B}), \omega) & \\ (T^*\mathcal{B}, \omega_1) \arrow[ur, "p_1^*"] & & (T^*\mathcal{B}, \omega_2) \arrow[ul, "p_2^*"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{B} \times \mathcal{B} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{B} & & \mathcal{B} \end{tikzcd}$$

or, alternatively, $T^*(\mathcal{B} \times \mathcal{B}) \to \mathcal{B} \times \mathcal{B}$ has fibres $\mathfrak{n}_1 \times \mathfrak{n}_2$ over $(\mathfrak{b}_1, \mathfrak{b}_2)$, so $T^*(\mathcal{B} \times \mathcal{B}) \cong \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$ where the map is given by

$$(\mathfrak{n}, \mathfrak{b}_1, \mathfrak{n}, \mathfrak{b}_2) \mapsto (\mathfrak{n}, \mathfrak{b}_1, -\mathfrak{n}, \mathfrak{b}_2)$$

4.3. Proposition. *([**CG00**] Prop 3.3.4) Z is the union of conormal bundles to G-orbits in* $\mathcal{B} \times \mathcal{B}$*.*

4.4. Definition. For $S \subseteq M$, $T_S^*M = \ker(T^*M|_S \to T^*S)$. For all $x \in S$, $T_x^*S = \{(x, \zeta) \in T_x^*M \mid \zeta(\alpha) = 0, \forall \alpha \in T_xS\}$.

To prove this proposition, recall that $\mathbb{W} \overset{1-1}{\leftrightarrow} \{G\text{-diagonal orbits in } \mathcal{B} \times \mathcal{B}\}$ given by $w \mapsto Y_w$.

Proof of Proposition. For Ω a G-orbit in $\mathcal{B} \times \mathcal{B}$ through $(\mathfrak{b}_1, \mathfrak{b}_2)$, $T_{Y_w}^*(\mathcal{B} \times \mathcal{B}) \subseteq Z$. Recall $\tilde{\mathcal{N}} \cong T^*\mathcal{B} \cong G \times_{\mathcal{B}} \mathfrak{n} = G \times_B \mathfrak{b}^\perp$ where $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. Then, for a $\zeta \in T_{Y_w}^*(\mathcal{B} \times \mathcal{B})$, $\zeta = (x_1, \mathfrak{b}_1, x_2, \mathfrak{b}_2) \in \mathfrak{g}^* \times \mathcal{B} \times \mathfrak{g}^* \times \mathcal{B}$ with $x_1 \in \mathfrak{b}_1^\perp, x_2 \in b_2^\perp$. Then, for any $\alpha \in T_{(\mathfrak{b}_1, \mathfrak{b}_2)\Omega}$, $\alpha = (u.\mathfrak{b}_1, u.\mathfrak{b}_2)$ for $u \in \mathfrak{g}$ and thus $\langle x_1, u \rangle + \langle x_2, u \rangle = 0$ for all $u \in \mathfrak{g}$. Thus, we get $x_1 = -x_2$. So, $\zeta = (x_1, \mathfrak{b}_1, -x_1, \mathfrak{b}_2) \in Z$. □

4.5. Corollary. *([**CG00**] Cor 3.3.5)*

$$Z = \bigsqcup_{w \in \mathbb{W}} T_{Y_w}^*(\mathcal{B} \times \mathcal{B})$$

Moreover, the irreducible components of Z are $T_{Y_w}(\mathcal{B} \times \mathcal{B})$.

4.6. Theorem. *Fix a Borel $\mathfrak{b} \subseteq \mathfrak{g}$ with $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ and Bsuch that Lie(B) $= \mathfrak{b}$ and $\mathfrak{b}^{\perp} = \mathfrak{n} \subseteq \mathfrak{g}$. Then, for any G-orbit $\mathcal{O} \subseteq \mathfrak{g}$ and $x \in \mathcal{O} \cap \mathfrak{b}$, we get $\mathcal{O} \cap (x + \mathfrak{n})$ is a lagrangian subvariety of $\mathcal{O}$.*

4.7. Definition. For a symplectic manifold M, $Z \subseteq M$ is *lagrangian subvariety* of M if, for all $x \in Z$, $T_x Z \subseteq T_x M$ is a lagrangian subspace of $T_x M$.

4.8. Definition. A *lagrangian subspace* W of symplectic space V is a subspace with $\dim W = \frac{1}{2} \dim V$ and on which the symplectic form is 0.

Proof Idea of Theorem. Focus on x nilpotent. □

4.9. Lemma. *([**CG00**] 3.3.8) $\dim(\mathcal{O} \cap \mathfrak{n}) \leq \frac{1}{2} \dim \mathcal{O}$, that is, $\mathcal{O} \cap \mathfrak{n}$ is isotropic.*

Proof. Let $n = \dim \mathfrak{n} = \dim \mathcal{B}$ so $\dim T^*\mathcal{B} = \dim Z = 2n$ by the identification of Z with a disjoint union of conormal spaces and defintion of a conormal bundle. Furthermore, let $Z_{\mathcal{O}} = \mu_Z^{-1}(\mathcal{O})$ for $\mathcal{O} \subseteq \mathcal{N}$, that is $\mathbb{Z}_{\mathcal{O}} = \{(x, \mathfrak{b}, \mathfrak{b}') \in Z \mid x \in \mathcal{O}\}$. Then, for any $x \in \mathcal{O}$, we have fibration

$$\begin{array}{ccc} \mathcal{B}_x \times \mathcal{B}_x & \longrightarrow & Z_{\mathcal{O}} \subseteq Z \\ \Big| & & \Big\downarrow \\ \mathcal{B}_x = \mu^{-1}(x) & & \mathcal{O} \end{array}$$

and so, by a subadditivity property of fibrations,

$$\dim \mathcal{O} + 2 \dim \mathcal{B}_x \leq \dim Z_{\mathcal{O}}.$$

Furthermore, because $Z_{\mathcal{O}} \subseteq Z$,

$$\dim Z_{\mathcal{O}} \leq \dim Z = 2n$$

Given $x \in \mathcal{O}$ and $x \in \mathfrak{n} \subseteq \mathfrak{b}$, we define

$$S := \{g \in G \mid g\mathfrak{b}g^{-1} \in \mathcal{B}_x\}$$

It turns out that S is B-stable and so we get isomorphism

$$B/S \xrightarrow{\sim} \mathcal{B}_x$$
$$Bg \mapsto g\mathfrak{b}g^{-1}$$

and thus

$$\dim \mathcal{O} + 2 \dim \mathcal{B}_x \leq 2n \Longrightarrow \dim S - \dim B + \frac{1}{2} \dim \mathcal{O} \leq n.$$

Note that $\mathcal{O} \cap \mathfrak{n} = \{gxg^{-1} \mid g \in S\}$. This leads us to the fact that

$$S/Z_G(x) \xrightarrow{\sim} \mathcal{O} \cap \mathfrak{n}$$

and so $\dim S - \dim Z_G(x) = \dim(\mathcal{O} \cap \mathfrak{n})$, and from there

$$\dim S + \frac{1}{2}\dim \mathcal{O} \leq n + \dim B = \dim G$$
$$\Longrightarrow \dim(\mathcal{O} \cap \mathfrak{n}) + \frac{1}{2}\dim \mathcal{O} \leq \dim G - \dim Z_G(x) = \dim \mathcal{O} \text{ by orbit-stabilizer.}$$

Thus, rearranging our inequality, we are done. □

Proof of Theorem. Given the lemma above, it suffices to show that $\mathcal{O} \cap \mathfrak{n}$ is coisotropic, that is $\dim(\mathcal{O} \cap \mathfrak{n}) \geq \frac{1}{2}\dim \mathcal{O}$. Consider

$$W \subseteq V \to W^{\perp\omega} \subseteq W$$

We can view $\mathcal{O}$ as a symplectic manifold with Hamiltonian B-action. This gets us a function $\mu_B \colon \mathcal{O} \to \mathfrak{b}^*$ and thus gives us the identification $\mathcal{O} \cap \mathfrak{n} = \mu_B^{-1}(0)$, which is coisotropic ([**CG00**] Thm 1.5.7). □

4.10. Example. Take $G = SL_n$. Then, let

$$\mathcal{O} = \{x \mid x \text{ nilpotent}, \operatorname{rank} x = 1\} \subseteq \mathfrak{sl}_n$$

Note that, in this case $x + \mathfrak{n} = x$ since x is nilpotent. If we take $V = \mathbb{C}^n$, then x is of the form $v \otimes w^* \colon V \to V$ given by $u \mapsto w^*(u)v$. The nilpotence of $x = v \otimes w^*$ tells us that $w^*(v) = \operatorname{tr}(x) = 0$. Thus, we get a surjection from the set

$$\{v \otimes w^* \mid w^*(v) = 0\} \twoheadrightarrow \mathcal{O} \text{ the orbit of trace free rank 1 linear maps.}$$

and thus $\dim \mathcal{O} = (2n-1) - 1 = 2n - 2$. In coordinates, this gives us

$$\mathcal{O} = \{x = (a_{ij}) \mid x \neq 0, a_{ij} = \alpha_i\beta_j, \sum \alpha_i\beta_j = 0\}$$

So, let $\mathfrak{n}$ be the set of upper triangular nilpotent matrices. Then,

$$\mathcal{O} \cap \mathfrak{n} = \{x = (a_{ij}) \mid x \neq 0, a_{ij} = 0 \text{ for } i > j, a_{ij} = \alpha_i\beta_j, \sum \alpha_i\beta_j = 0\}$$

Thus, irreducible components of $\mathcal{O} \cap \mathfrak{n}$ are

$$\left\{ \begin{pmatrix} 0 & & & a_{1,k+1} & \cdots & a_{1,n} \\ & 0 & & \vdots & & \vdots \\ & & 0 & a_{k,k+1} & \cdots & a_{k,n} \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \;\middle|\; \text{rows are all proportional} \right\}$$

Therefore, since all the rows are proportional, there are only $n-1$ choices for entries, so it has dimension $n-1$ and is thus a Lagrangian subspace. So, we have found $\mathcal{O} \cap (x + \mathfrak{n}) = \mathcal{O} \cap \mathfrak{n}$ as a lagrangian subvariety.

4.11. Example. Consider $\mathcal{O}_x$ for $x \in \mathfrak{g}^{sr}$. So, $\mathcal{O} \cong G/T$. Take $\mathfrak{b}$ such that $x \in \mathfrak{b}$, $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ and B such that $\operatorname{Lie}(B) = \mathfrak{b}$. Then, $x \in \mathfrak{b} \cap \mathfrak{g}^{sr}$ so

$x+\mathfrak{n} = Bx \subseteq \mathcal{O}$ is a single B-orbit and, according to our theorem, $x+\mathfrak{n}\cap\mathcal{O}$ is a Lagrangian subvariety of $\mathcal{O}$. To see this, we note that

$$\dim(\mathcal{O}\cap(x+\mathfrak{n})) = \dim(x+\mathbb{N}) = \dim\mathfrak{n} = \frac{1}{2}\dim G/T = \frac{1}{2}\dim\mathcal{O}$$

Now, taking $\tilde{\mathcal{O}} = \mu^{-1}(\mathcal{O})$, we get that $\mu\colon\tilde{\mathcal{O}}\to\mathcal{O}$ is an unramified cover with $\#W$ leaves. Note $\pi_1(\mathcal{O}) = 0$. Thus, we get

$$Z_{\mathcal{O}} = \mu_Z^{-1}(\mathcal{O}) \cong \tilde{\mathcal{O}}\times_{\mathcal{O}}\tilde{\mathcal{O}}$$

Using a resolution, we get

$$m := 2\dim\mathfrak{n} = 2\dim\mathcal{B} = \dim\mathcal{N}$$

What was the last part of this example supposed to show?

4.12. Corollary. *The irreducible components of $Z_{\mathcal{O}}$ have the same dimension*

$$\dim Z_{\mathcal{O}} = \dim Z = m$$

Understand this proof better.

Proof. Recall $\pi\colon\tilde{\mathcal{N}}\cong G\times_B\mathfrak{n}\to\mathcal{B}\cong G/B$ which restricts to the isomorphism $\tilde{\mathcal{O}}\cong G\times_B(\mathcal{O}\cap\mathfrak{n})$. This is a fibration and the dimension of the fibres is $\frac{1}{2}\dim\mathcal{O}$. Irreducible components of $\tilde{\mathcal{O}}$ have dimension $\dim(G/B)+\frac{1}{2}\dim\mathcal{O}$. This gives us that irreducible components of $Z_{\mathcal{O}}$ have dimension

$$2\dim\tilde{\mathcal{O}} - \dim\mathcal{O} = 2(\dim G/B + \frac{1}{2}\dim\mathcal{O}) - \dim\mathcal{O} = 2\dim G/B = m$$

$\square$

4.13. Remark. Later, we will use this to show $H_m(Z;\mathbb{Q})\cong\mathbb{Q}[\mathbb{W}]$.

4.14. Corollary. *$Z = \bigsqcup_{\mathcal{O}} Z_{\mathcal{O}}$ is a partition of Z into locally closed subsets of dimension* $\dim Z$.

4.15. Corollary (Robin Schensted Correspondance). *The number of nilpotent orbits of $\mathfrak{g}$ is finite.*

Understand this proof better.

Proof. Let us examine the irreducible components of $Z_{\mathcal{O}}$. Let $x\in\mathcal{O}$. Then, $\mathcal{O} = G/G_x$ by orbit stabilizer. Furthermore, G_x acts on $\mathcal{B}_x$, giving us a G-equivariant isomorphism $\tilde{O}\cong G\times_{G_x}\mathcal{B}_x$.

$$\begin{tikzcd}
\tilde{\mathcal{O}} \arrow[d, "\mu"'] \arrow[r, "\sim"] & G\times_{G_x}\mathcal{B}_x \arrow[d] \\
\mathcal{O} & G/G_x
\end{tikzcd}$$

So, we get $Z_{\mathcal{O}} = \tilde{\mathcal{O}}\times_{\mathcal{O}}\tilde{\mathcal{O}}\cong G\times_{G_x}(\mathcal{B}_x\times\mathcal{B}_x)$, which allows us to see that the irreducible components of $Z_{\mathcal{O}}$ are $G\times_{G_x}(\mathcal{B}_1\times\mathcal{B}_2)$ where $\mathcal{B}_1,\mathcal{B}_2$ are irreducible components of $\mathcal{B}_x$. Thus,

$$\dim\mathcal{O}+\dim\mathcal{B}_1+\dim\mathcal{B}_2 = 2\dim\mathcal{B}$$

and if we take $\mathcal{B}_1 = \mathcal{B}_2$, we get

$$\dim \mathcal{B}_x = \dim \mathcal{B} - \frac{1}{2}\dim \mathcal{O}$$

which is the dimension of irreducible points of $\mathcal{B}_x$. Thus, there is a one-to-one correspondance

$$\{\text{Irreducible components of } Z_{\mathcal{O}}\} \xleftrightarrow{\text{1-1}} \{G_x\text{-orbits on pairs } (\mathcal{B}_x^\alpha, \mathcal{B}_x^\beta)\}$$

where $C_x = G_x/(G_x^\circ)$ and C_x acts on $\{\mathcal{B}_x^\alpha\}$. □

(9/18/2017) Lecture 4: Lagrangian Construction of the Weyl Group (presented by Thomas Sale).

5. The Lagrangian Construction of the Weyl Group

5.1. Definition. Throughout this talk, let

$$m := 2\dim_{\mathbb{R}} \mathfrak{n} = \dim_{\mathbb{R}} Z = \frac{1}{2}\dim_{\mathbb{R}}(\tilde{\mathcal{N}} \times \tilde{\mathcal{N}})$$

Now, we must review some material

5.0.1. *Borel-Moore Homology ([**CG00**] 2.6).*

5.2. Definition. We define the *Borel-Moore Homology* to be

$$H_*^{BM}(X) = H_*(\hat{X}, \{\infty\})$$

where $\hat{X} = X \cup \{\infty\}$ is the 1-point compactification of X.

5.3. Remark. Note that $H_*^{BM}(X) = H_*(\overline{X}, \overline{X}\setminus X)$ for $\overline{X}$ any compacticiation of X.

5.4. Proposition. *Let M be a smooth not-necessarily compact oriented manifold and $X \subseteq M$ with $\dim_{\mathbb{R}} M = N$. Then,*

$$H_i^{BM}(X) \cong H^{N-i}(M, M \setminus X)$$

5.5. Definition. Let us define the *intersection pairing* to be a billinear pairing

$$\begin{tikzcd}
H_i^{BM}(Z) \times H_j^{BM}(\tilde{Z}) \arrow[r, "\cap"] \arrow[d, "\sim"] & H_{i+j-N}^{BM}(Z \cap \tilde{Z}) \arrow[d, "\sim"] \\
H^{N-i}(M, M \setminus Z) \times H^{N-j}(M, M \setminus \tilde{Z}) \arrow[r, "\cup"] & H^{2N-j-i}(M, (M \setminus Z) \cup (M \setminus \tilde{Z}))
\end{tikzcd}$$

where $\cup$ is the standard cup product on cohomology and $Z, \tilde{Z}$ are closed subspaces of M.

5.6. Definition. We define the *Kunneth formula* for the Borel-Moore homology via the Kunneth formula for standard homology.

$$\begin{array}{ccc} H_*^{BM}(M_1) \times H_*^{BM}(M_2) & \xrightarrow{\boxtimes} & H_*^{BM}(M_1 \times M_2) \\ \downarrow\sim & & \downarrow\sim \\ H_*(\overline{M_1}, \overline{M_1} \setminus M_1) \times H_*(\overline{M_2}, \overline{M_2} \setminus M_2) & \xrightarrow{\boxtimes} & H_*(M_1 \times M_2, \overline{M_1 \times M_2} \setminus (\overline{M_1} \times M_2 \cup M_1 \times \overline{M_2})) \end{array}$$

5.1. Set-Theoretic Composition ([CG00] 2.7).

5.7. Definition. Let $Z_{12} \subseteq M_1 \times M_2$ and $Z_{23} \subseteq M_2 \times M_3$. Then, we define

$$Z_{12} \circ Z_{23} := \{(m_1, m_3) \in M_1 \times M_3 \mid \exists m_2 \in M_2 \text{ such that } (m_1, m_2) \in Z_{12} \text{ and } (m_2, m_3) \in Z_{23}\}$$

Now, we can define convolution products on Borel-Moore homology with $d = \dim_{\mathbb{R}} M_2$. **Note that, from this point onward, we are dropping the BM and all homology is Borel-Moore homology.**

5.8. Definition. We define the *convolution product* to be a map

$$*\colon H_1(Z_{12}) \times H_j(Z_{23}) \to H_{i+j-d}(Z_{12} \circ Z_{23})$$
$$(c_{12}, c_{23}) \mapsto c_{12} * c_{23}$$

where

$$c_{12} * c_{23} = (p_{13})_*((c_{12} \boxtimes [M_3]) \cap ([M_1] \boxtimes c_{23}))$$

and $p_{13}\colon M_1 \times M_2 \times M_3 \to M_1 \times M_3$ the standard projection map that is also proper.

5.9. Example. Let f, g be smooth functions $f\colon M_1 \to M_2$ and $g\colon M_2 \to M_3$. Then,

$$\text{Graph}(f) \circ \text{Graph}(g) = \text{Graph}(g \circ f)$$

Thus,

$$[\text{Graph}(f)] * [\text{Graph}(g)] = [\text{Graph}(g \circ f)]$$

5.2. Lagrangian Construction of the Weyl Group.

5.10. Definition. For the Steinberg variety $Z \subseteq \tilde{N} \times \tilde{N}$, we define

$$H(Z) := H_m(Z, \mathbb{Q}), \text{ the top Borel-Moore homology}$$

where $m = \dim_{\mathbb{R}}(Z) = \dim_{\mathbb{R}}(\tilde{N})$, as above.

5.11. Remark. Note that $H(Z)$ is an algebra with respect to the convolution $*$ since $Z \circ Z = Z$. Take $M_1 = M_2 = M_3 = \tilde{N}$ to get

$$*\colon H_m(Z) \times H_m(X) \to H_{2m-m}(Z \circ Z)$$

5.12. Proposition. $\dim H(Z) = |\mathbb{W}|$

Proof. Recall that $Z = \sqcup Z_{\mathcal{O}}$ (4.14). The number of these subsets is precisely $|\mathbb{W}|$. This gives the desired result. $\square$

5.13. THEOREM. *There is a canonical algebra isomorphism*

$$H(Z) \cong \mathbb{Q}[\mathbb{W}]$$

To prove this theorem, we must give a few definitions and lemmas. Let us fix $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$, $w \in \mathbb{W}$, and regular semisimple $h \in \mathfrak{b}$.

5.14. DEFINITION. Consider subset $S \subseteq \mathfrak{h}$. We define

$$\tilde{\mathfrak{g}}^S := \nu^{-1}(S) = G \times_B (S + \mathfrak{n})$$

5.15. EXAMPLE. Note that

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^{\mathfrak{h}} = G \times_B (h + \mathfrak{n}) = G \times_B \mathfrak{b}$$

and

$$\tilde{\mathfrak{g}}^0 = G \times_B \mathfrak{n} = \tilde{\mathcal{N}}$$

5.16. DEFINITION. We define $\Lambda_w^h \subseteq \tilde{g}^{w(h)} \times \tilde{g}^h$ to be the graph of the $\mathbb{W}$-action on $\tilde{g}^{sr}$. Thus,

$$\Lambda_w^h = \{(x, \mathfrak{b}, x, \mathfrak{b}') \mid x \in \mathfrak{b} \cap \mathfrak{b}', \nu(x, \mathfrak{b}') = h, (\mathfrak{b}, \mathfrak{b}') \in Y_w\} \subseteq \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$$

5.17. PROPOSITION. *The projection map $\pi\colon \tilde{\mathfrak{g}} \to \mathcal{B}$ given by $(x, \mathfrak{b}) \mapsto \mathfrak{b}$ induces a map $\pi^2\colon \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \mathcal{B} \times cB$ given by $\pi^2(\Lambda_w^h) = Y_w$*

PROOF. Note that

$$\Lambda_w^h \hookrightarrow \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \overset{\pi\times\pi}{\to} \mathcal{B} \times \mathcal{B}$$

This image is G-stable with respect to the diagonal action and, by the set characterization of Λ_w^h, $\pi^2(\Lambda_w^h) = Y_w$. □

5.18. LEMMA. *The fibrations $\pi^2\colon \Lambda_w^h \to Y_w$ and $G \times_{B\cap w(B)} (h + \mathfrak{n} \cap w(\mathfrak{n})) \to G/(B \cap w(B))$ are equivalent.* why?

PROOF. Identify $G/B \times G/w(B)$ with $\mathcal{B} \times \mathcal{B}$ via the assignment

$$(g_1B, g_2wB) \mapsto (g_1\mathfrak{b}g_1^{-1}, g_2w(\mathfrak{b})g_2^{-1})$$

□ Finish this?

5.19. LEMMA. *For $w, y \in \mathbb{W}$,*

$$[\Lambda_y^{w(h)}] * [\Lambda_w^h] = [\Lambda_{yw}^h]$$

PROOF. Since Λ_w^h is a graph, then, for $w, y \in \mathbb{W}$,

$$\Lambda_y^{w(h)} \circ \Lambda_w^h = \Lambda_{yw}^h$$

Thus, we get the desired result. □

5.20. DEFINITION. We define

$$\mathfrak{H}_w := \text{Graph}(\mathfrak{H} \overset{w}{\to} \mathfrak{H}) \subseteq \mathfrak{H} \times \mathfrak{H}$$

$$\tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}} := \{(y, x) \in \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \mid \nu(y) = w(\nu(y))\}$$

$$\nu_w := \nu \times \nu|_{\tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}}}$$

and

$$\Lambda_w := (\tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}}) \cap (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$$

5.21. Remark. Note that $\nu_w^{-1}(0) = \nu^{-1}(0) \times \nu^{-1}(0) = \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$, the graph over special point 0. Also note that

$$\begin{aligned} \Lambda_w \cap \nu_w^{-1}(0) &\subseteq (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) \cap (\tilde{\mathcal{N}} \times \tilde{\mathcal{N}}) \\ &\subseteq Z \end{aligned}$$

5.22. Definition. We define

$$\mathfrak{H}_w^{reg} := \mathrm{Graph}(\mathfrak{H}^{reg} \xrightarrow{w} \mathfrak{H}^{reg})$$

and thus

$$\begin{aligned} \Lambda_w^{reg} &:= \Lambda_2 \cap \nu_w^{-1}(\mathfrak{H}_w^{reg}) \\ &= (\tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}}) \cap (\tilde{g}^{sr} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}^{sr}) \\ &= \mathrm{Graph}(\tilde{g}^{sr} \xrightarrow{w} \tilde{g}^{sr}) \end{aligned}$$

Now, we seek to specialize $[\Lambda_w^{reg}]$ at $0 \in \mathfrak{H}$.

5.23. Remark. Note that $\mathfrak{H} \backslash \mathfrak{H}_{reg}$ is the root hyperplanes and $\mathrm{codim}_{\mathbb{R}}(\mathfrak{H} \backslash \mathfrak{H}_{reg}) = 2$.

5.24. Definition. Let $\ell \subseteq \mathfrak{H}^{reg}$ be a real vector space with dimension less than or equal to 2. Then, we define

$$\ell^* := \ell \setminus \{0\} = \ell \cap \mathfrak{H}^{reg}$$

5.25. Proposition. *The following diagrams commute*

$$\begin{tikzcd} \tilde{\mathfrak{g}} \times_{\mathfrak{H}_w} \tilde{\mathfrak{g}} \arrow[r, hook] \arrow[d, "\nu_w"] & \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \arrow[d, "\nu \times \nu"] \\ \mathrm{Graph}(\mathfrak{H} \xrightarrow{w} \mathfrak{H}) \arrow[r, hook] & \mathfrak{H} \times \mathfrak{H} \end{tikzcd} \rightsquigarrow \begin{tikzcd} \tilde{\mathfrak{g}}^{w(\ell)} \times_{\ell_w} \tilde{\mathfrak{g}}^{\ell} \arrow[r, hook] \arrow[d, "\nu_w"] & \tilde{\mathfrak{g}}^{w(\ell)} \times \tilde{\mathfrak{g}}^{\ell} \arrow[d, "\nu \times \nu"] \\ \mathrm{Graph}(\ell \xrightarrow{w} w(\ell)) \arrow[r, hook] & \ell \times w(\ell) \end{tikzcd}$$

Proof. The first diagram follows from the definition of ν_w. The second diagram follows from the natural projection $\tilde{\mathfrak{g}} \to \ell$ from the definition of ν_w in 5.20. $\square$

5.3. Specialization. In our case, specialization will be the process

This entire specialization stuff makes no sense to me.

$$\begin{aligned} H_*(\Lambda_w^{\ell}) &\to H_*(Z) \\ [\Lambda_w^{\ell^*}] &\mapsto [\Lambda_w^{0 \cdot h}] \in H_m(Z) \end{aligned}$$

5.26. Lemma. *(*[**CG00**] *3.4.11)* $[\Lambda_w^{0 \cdot h}]$ *does not depend on* h.

Proof. The proof follows from the transitivity of specialization. See [**CG00**] 2.6.38. The specialization at ℓ equals the specialization at $\mathbb{R}h$ for any $h \in \ell^*$. For $h, h' \in \mathfrak{H}^{ref}$, draw a polygonal path in $\mathfrak{H}^{reg}$ from h to h' with vertices $h = h_1, h_2, \ldots, h_m = h'$. Then, the specialization at $\mathrm{span}_{\mathbb{R}}(h_1, h_{i+1})$ equals the specialization at $\mathbb{R}h_1$ also equals the specialization at $\mathbb{R}h_{i+1}$. $\square$

5.27. Lemma. *Specialization commutes with convolution. (See* [**CG00**] *2.7.23)*

5.28. Lemma.

$$\lim_{h\to 0}([\Lambda^h_{yw} = [\Lambda^{w(h)}_y] * [\Lambda^h_w]]) = ([\Lambda^0_{yw} = [\Lambda^0_y] * [\Lambda^0_w]])$$

5.29. Proposition. *$\{[\Lambda^0_w] \mid w \in \mathbb{W}\}$ is a basis of $H(Z)$.*

Proof. $H(Z)$ has basis

$$\{T^*_w = \overline{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})} \mid w \in \mathbb{W}\}.$$

So, let

$$\Lambda_Y = \sum_{w\in\mathbb{W}} n_{yw} T^*_w$$

Now, we have

$$\begin{array}{c} \pi^2(\Lambda^h_y) \subseteq Y_y \\ \Big\downarrow {\scriptstyle \lim_{h\to 0}} \\ \pi^2(\Lambda^0_y) \subseteq \overline{Y_y} \end{array}$$

thus implying that $[\Lambda^0_y]$ involves only T^*_w such that $Y_w \subseteq \overline{Y_y}$, which leads to the Bruhat order. $\square$

This is a mess! Understand it better

5.30. Lemma. *$n_{ww} = 1$ for any $w \in \mathbb{W}$.*

Proof. Over an open subset $U \subseteq \mathcal{B}\times\mathcal{B}$ containing Y_w,

$$\begin{tikzcd} \Lambda^h_w = G \times_{B\cap w(B)} (h + \mathfrak{n}\cap w(\mathfrak{n})) \arrow[r, "\lim_{h\to 0}"] \arrow[d] & G \times_{B\cap w(B)} (\mathfrak{n}\cap w(\mathfrak{n})) \arrow[d] \\ Y_w & Y_w \end{tikzcd}$$

Thus, $n_{ww} = 1$. $\square$

5.31. Remark. Kashiwara and Saito showed that the n_{yw} are not the Kazhdan-Lusztig numbers.

(9/25/2017) Lecture 5 (presented by Ramanujan Santharoubane).

6. Geometric Analysis of $H(Z)$

Previously, for the Springer variety Z, we have shown that $H(Z) \cong \mathbb{Q}(\mathbb{W})$ and $\mathbb{C}\otimes_{\mathbb{Q}} H(Z) = \mathbb{C}[\mathbb{W}]$ is a semisimple Lie algebra. By the Artin-Wedderburn theorem, we have

$$\mathbb{C}\otimes_{\mathbb{Q}} H(Z) = \bigoplus_\alpha \mathrm{End}_{\mathbb{C}}(E_\alpha)$$

where $\{E_\alpha\}$ is a complete set of $\mathfrak{g}$ irreducible representations. Our goal for this section is to understand this decomposition.

Recall that, for the Springer resolution

$$\begin{tikzcd} \tilde{\mathcal{N}} \arrow[d, "\mu"] \\ \mathcal{N} \end{tikzcd} \qquad\qquad Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$$

and, for $Y \subseteq \mathcal{N}$,

$$Z_y = \mu^{-1}(Y) \times_Y \mu^{-1}(Y) \subseteq \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$$

Now, $Z_Y \circ Z = Z_Y = Z \circ Z_Y$.

6.1. Proposition. *$H(Z_Y)$ has a $H(Z)$-bimodule structure.*

(10/9/2017) Lecture 6 (presented by Liron Speyer).

(10/9/2017) Lecture 7 (presented by George H. Seelinger).

7. Applications of the Jacobson-Morozov Theorem

7.1. Statement and Immediate Applications.

7.1. Definition. Let (e, f, h) be a triple of elements in some semisimple Lie algebra $\mathfrak{g}$ over a field of characteristic 0. We say (e, f, h) is a $\mathfrak{sl}_2$-*triple* if

$$[h, e] = 2e, \ [h, f] = -2f, \ [e, f] = h.$$

We recall from a first course in semisimple Lie algebras the following proposition.

7.2. Proposition. *A Lie algebra $\mathfrak{g}$ decomposes into a direct sum of finite-dimensional subspaces, each of which is isomorphic to V_j, the $j+1$-dimensional simple $\mathfrak{sl}_2$-module with highest weight j.* Elaborate

To each of these simple $\mathfrak{sl}_2$-modules, we can associate an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. However, the Jacobson-Morozov Theorem can provide us with a partial converse to this fact.

7.3. Theorem (Jacboson-Morozov Theorem). [**CG00**, 3.7.1] *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. For any nilpotent element $e \in \mathfrak{g}$, there exists $h, f \in \mathfrak{g}$ such that $\{e, f, h\}$ is an $\mathfrak{sl}_2$-triple.*

Thus, there exists a Lie algebra homomorphism $\gamma \colon \mathfrak{sl}_2 \to \mathfrak{g}$ such that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto e, \ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto f, \ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto h$$

Moreover, h is a semisimple element and f is a nilpotent element of $\mathfrak{g}$.

7.4. Remark. This triple (e, f, h) associated with e is not necessarily unique. The following proposition, due to Kostant, tells us to what extent the associated triple is unique.

7.5. Proposition (Kostant's Theorem). [**CG00**, 3.7.3] *Let $Z_G(e)$ denote the centralizer in G of e. Then, the above homomorphism $\gamma\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ is determined uniquely up to conjugation by an element in the unipotent radical of the group $Z_G(e)$.*

Proof of Jacobson-Morozov Theorem when $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. [**CG00**, p 184] By the Jordan canonical form, any nilpotent element is conjugate to a direct sum of Jordan blocks. Thus, we need only show the theorem is true when e is a single m by m Jordan block. Since e is nilpotent, all of its eigenvalues are 0, so

$$e = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

Then, we take

$$h = \begin{pmatrix} m-1 & 0 & \cdots & 0 \\ 0 & m-3 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & -m+1 \end{pmatrix}$$

$$f = \begin{pmatrix} 0 & 0 & & & & \\ m-1 & 0 & 0 & & & \\ 0 & 2(m-2) & 0 & & & \\ & & \ddots & & & \\ & & & -2(m-2) & 0 & 0 \\ & & & & -(m-1) & 0 \end{pmatrix}$$

From this, it is straightforward to check that the set (e, f, h) actually forms an $\mathfrak{sl}_2$-triple. $\square$

7.6. Lemma. [**CG00**, 3.2.16] *$e \in \mathfrak{g}$ is nilpotent if and only if*

$$(e, x) = \mathrm{tr}(\mathrm{ad}_e \cdot \mathrm{ad}_x) = 0 \text{ for all } x \in Z_{\mathfrak{g}}(e)$$

Proof. ($\Longrightarrow$). Assume e is nilpotent and $x \in Z_{\mathfrak{g}}(e)$. Since ad_e and ad_x commute , for large enough k, why?

$$(\mathrm{ad}_e \cdot \mathrm{ad}_x)^k = \mathrm{ad}_e^k \cdot \mathrm{ad}_x^k = 0$$

by the nilpotency of x. Thus, $\mathrm{ad}_e \cdot \mathrm{ad}_x$ is also nilpotent and thus has trace 0.

($\Longleftarrow$). Now assume $(e, x) = 0$ for all $x \in Z_{\mathfrak{g}}(e)$. We note that ad_e is skew-symmetric with respect to the Killing form and so $\mathrm{im}(\mathrm{ad}_e) = \ker(\mathrm{ad}_e)^{\perp}$, where $\perp$ stands for the annihilator with respect to $(,)$. Thus, we have

7.7. Lemma. *$e \in \mathrm{im}(\mathrm{ad}_e)$ is equivalent to $(e, Z_{\mathfrak{g}}(e)) = 0$ is equivalent to there exists an element $h \in \mathfrak{g}$ such that $[h, e] = e$.*

However, we can also show there exists a *semisimple* element $s \in \mathfrak{g}$ such that $[s, e] = e$. To do this, take h as above and write its Jordan decomposition $h = s + n$ where s semisimple and n nilpotent. Since e is an eigenvector for ad_h, then e is an eigenvector for ad_s and ad_n. However, ad_n is nilpotent and thus only has eigenvalues 0, so $\mathrm{ad}_n e = 0$. Therefore, $\mathrm{ad}_s e = \mathrm{ad}_h e = e$ and so we have found semisimple $s \in \mathfrak{g}$ such that $[s, e] = e$.

So, take $h \in \mathfrak{g}$ to be semisimple such that $[h, e] = e$. Then, we can decompose $\mathfrak{g}$ into ad_h-eigenspaces:

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{C}} \mathfrak{g}_\alpha, \;\; \mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid \mathrm{ad}_h(x) = \alpha \cdot x\}$$

where $h \in \mathfrak{g}_0$ and $e \in \mathfrak{g}_1$. We also have relation $\mathrm{ad}_h \circ \mathrm{ad}_e = \mathrm{ad}_e \circ (1 + \mathrm{ad}_h)$ which tells us ad_e takes $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{\alpha+1}$. Thus, ad_e^k takes $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{\alpha+k}$ and since there are only finitely many non-zero spaces $\mathfrak{g}_\alpha$, we get $\mathrm{ad}^k e = 0$ for sufficiently large k. Thus, e is nilpotent. $\square$

7.8. Lemma. *If $e \in \mathfrak{g}$ is nilpotent, then there exists a semisimple $h \in \mathfrak{g}$ such that $[h, e] = e$.*

Proof. If $e \in \mathfrak{g}$ is nilpotent, then by the lemma above, $(e, Z_{\mathfrak{g}}(e)) = 0$. Thus, following the "only if" part of the proof above, we can find such an h. $\square$

Proof of Full Theorem. [**CG00**, pp191–2] Fix a nilpotent $e \in \mathfrak{g}$. Note that if $\dim \mathfrak{g} = 3$, then we are done since $\mathfrak{g} = \mathfrak{sl}_2$.

(1) Decomposing a non-nilpotent $x \in Z_{\mathfrak{g}}(e)$ (assuming such an x exists) into its Jordan decomposition $x = s + n$ and using $[x, e] = 0 \Longrightarrow [s, e] = 0$ (see [**Hum72**, p]), we argue by induction on $\dim \mathfrak{g}$ to reduce to the case where the subalgebra $Z_{\mathfrak{g}}(e)$ consists of nilpotent elements only. Since $[s, e] = 0$, we have $s \in Z_{\mathfrak{g}}(e)$, so $Z_{\mathfrak{g}}(e)$ contains a non-zero semisimple element. Furthermore, the centralizer of such an element is a *proper* reductive Lie subalgebra of $\mathfrak{t} \subseteq \mathfrak{g}$. Since e commutes with the semisimple component of $\mathfrak{t}$, we have $e \in \mathfrak{t}$. Thus, this semisimple component of $\mathfrak{t}$ is a Lie algebra of dimension less than $\dim \mathfrak{g}$ containing e, so we are done by the inductive hypothesis. So, we must deal with the case of $Z_{\mathfrak{g}}(e)$ consisting only of nilpotent elements.

Fill in page number

(2) Show there is a semisimple $h \in \mathfrak{g}$ such that $[h, e] = 2e$. By the lemma above, for any nilpotent element $e \in \mathfrak{g}$, we have a semisimple $h \in \mathfrak{g}$ such that $[h, e] = e \implies [2h, e] = 2e$.
(3) Fix an $h \in \mathfrak{g}$ such that $[h, e] = 2e$ as above and introduce weight space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{C}} \mathfrak{g}_\alpha, \ \mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid \mathrm{ad}_h(x) = \alpha \cdot x\}.$$

where $h \in \mathfrak{g}_0, e \in \mathfrak{g}_2$ and $\mathrm{ad}_h \circ \mathrm{ad}_e = \mathrm{ad}_e \circ (2 + \mathrm{ad}_h) \implies \mathrm{ad}_e$ takes $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{\alpha+2}$. Thus, if we find $f \in \mathfrak{g}_{-2}$ such that $\mathrm{ad}_e(f) = h$, we are done. Since ad_e shifts the gradation by 2, this amounts to showing that $h \in \mathrm{im}(\mathrm{ad}_e)$. However, from the lemma in the proof above, we see this holds if and only if $(h, Z_{\mathfrak{g}}(e)) = 0$. To prove this, we use the Jacobi identity and the fact that $[h, e] = 2e$ to get that, for $x \in Z_{\mathfrak{g}}(e)$,

$$\begin{aligned}
[e, [h, x]] &= -[h, [x, e]] - [x, [e, h]] \\
&= -[h, 0] + [x, 2e] \\
&= 0
\end{aligned}$$

and so $[h, Z_{\mathfrak{g}}(e)] \subseteq Z_{\mathfrak{g}}(e)$. So, we have that $\mathbb{C} \cdot h + Z_{\mathfrak{g}}(e)$ is a Lie subalgebra of $\mathfrak{g}$. Now, by the first part of the proof, we can assume x is nilpotent and using Engel's theorem, this tells us that $Z_{\mathfrak{g}}(e)$ is nilpotent and thus $\mathbb{C} \cdot h + Z_{\mathfrak{g}}(e)$ is a solvable Lie algebra. Thus, by Lie's theorem, for all $x \in \mathbb{C} \cdot h + Z_{\mathfrak{g}}(e)$, we can put ad_x in the upper triangular form so that, for $x \in Z_{\mathfrak{g}}(e)$, ad_x is strictly upper triangular. Therefore, for any $x \in Z_{\mathfrak{g}}(e)$, $\mathrm{ad}_h \cdot \mathrm{ad}_x$ is strictly upper triangular and so $\mathrm{tr}(\mathrm{ad}_h \cdot \mathrm{ad}_x) = 0 \implies (h, Z_{\mathfrak{g}}(e)) = 0 \implies h \in \mathrm{im}(\mathrm{ad}_e)$ gives us the existence of an $f \in \mathfrak{g}_{-2}$ such that $\mathrm{ad}_e(f) = h$.

□

7.9. Remark. In [**CM93**], parts (1) and (2) of the proof are morally equivalent. However, for part (3) in [**CM93**, pp 38–39], they proceed by contradiction, but follow a somewhat similar proof. In [**CG00**, p 190], they claim that the standard proof is quite elementary but involves several tricky lemmas. However, this proof made no use of anything terribly sophisticated in geometry, so I do not quite understand this statement.

7.10. Corollary. *Given a nilpotent $e \in \mathfrak{g}$, there exists a rational homomorphism $\gamma \colon SL_2(\mathbb{C}) \to G$ such that its differential $d\gamma \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ sends $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to e.*

Proof. $SL_2(\mathbb{C})$ is simply connected. Thus, any Lie algebra homomorphism can be extended to a (unique) Lie group homomorphism [1]. □

[1]This theorem is a highly non-trivial differential geometry proof, but proved in [**Hal03**, section 3.6]

Recall that any simple finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-module V looks like

$$\text{highest weight} \qquad \cdot \underset{e}{\overset{f}{\rightleftarrows}} \cdot \underset{e}{\overset{f}{\rightleftarrows}} \cdots \underset{e}{\overset{f}{\rightleftarrows}} \cdot \qquad \text{lowest weight}$$

where the dots correspond to h-eigenspaces, each of dimension 1, and $V = \ker f \oplus \operatorname{im} e$ where $\ker f$ is the rightmost vertex. This decomposition holds for any, not necessarily simple, $\mathfrak{sl}_2$-module.

7.11. Corollary. *Let the $\mathfrak{sl}_2$-triple (e, f, h) act on a finite dimensional vector space V. Assume that $v \in V$ is such that $f \cdot v = 0$ and $h \cdot v = -m \cdot v$. Then, m is a non-negative integer and we have $e^{m+1} \cdot v = 0$.*

Proof. This follows since every $\mathfrak{sl}_2(\mathbb{C})$-module is symmetric about the 0-eigenspace. $\square$

7.12. Corollary. *Any nilpotent element of a semisimple Lie algebra $\mathfrak{g}$ is acting as a nilpotent operator on any finite dimensional $\mathfrak{g}$-module.*

Proof. By the Jacobson-Morosov Theorem, any nilpotent element e is part of an $\mathfrak{sl}_2$-triple and thus, by the corollary above, for each $v \in V$, there is an $m \in \mathbb{N}$ such that $e^{m+1} \cdot v = 0$. Thus, since V is finite-dimensional, e is a nilpotent operator on V. $\square$

7.13. Corollary. *Fix a nilpotent $e \in \mathfrak{g}$ and corresponding $\mathfrak{sl}_2$-triple (e, f, h). Then,*

(a) All the eigenvalues of the operator $\mathrm{ad}_h \colon Z_{\mathfrak{g}}(e) \to Z_{\mathfrak{g}}(e)$ are non-negative integers.
(b) If all the eigenvalues of the operator $\mathrm{ad}_h \colon \mathfrak{g} \to \mathfrak{g}$ are even, then $\dim Z_{\mathfrak{g}}(e) = \dim Z_{\mathfrak{g}}(h)$.

Proof. From the $\mathfrak{sl}_2$-triple, we have an embedding $\mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}$. The adjoint action on $\mathfrak{g}$ of the image of the embedding makes $\mathfrak{g}$ an $\mathfrak{sl}_2(\mathbb{C})$-module. Then, by the decomposition of $\mathfrak{sl}_2(\mathbb{C})$-modules, we see that all the eigenvalues of $\mathrm{ad}_h \colon \mathfrak{g} \to \mathfrak{g}$ are integers and the weight space decomposition into ad_h-eigenspaces yields a $\mathbb{Z}$-grading on the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \ \ [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}, \forall i, j \in \mathbb{Z}$$

Furthermore, by the Jacobi identity, $Z_{\mathfrak{g}}(e)$ is ad_h-stable ($[e, [h, x]] = -[h, [x, e]] - [x, [e, h]] = -[h, 0] - [x, 2e] = 0$), so we have

$$Z_{\mathfrak{g}}(e) = \bigoplus_{i \in \mathbb{Z}} Z_{\mathfrak{g}_i}(e), \ \ Z_{\mathfrak{g}_i}(e) = Z_{\mathfrak{g}}(e) \cap \mathfrak{g}_i$$

Finally, since $\ker(e) = Z_{\mathfrak{g}}(e)$, then $Z_{\mathfrak{g}}(e) \cap \operatorname{im}(f) = \{0\}$, so $Z_{\mathfrak{g}}$ contains only highest weight vectors. From the general theory, we know that if v_0 is a highest weight vector, then $h.v_0 = [\dim(\operatorname{span}\{v_0, f.v_0, f^2.v_0, \ldots\}) - 1]v_0$, but such an eigenvalue must be a nonnegative integer. If all the eigenvalues are even, then repeated application of f to each element of $Z_{\mathfrak{g}}(e)$ will land it in $\mathfrak{g}_0 = Z_{\mathfrak{g}}(h)$ since $f\colon \mathfrak{g}_m \to \mathfrak{g}_{m-2}$ for all $m \in \mathbb{Z}$. $\square$

7.14. Theorem. *([**CG00**] cor 3.2.13 and thm 3.7.13). Let $e_1, \dots, e_n$ be a set of root vectors in $\mathfrak{n}$ for $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \oplus \mathfrak{n}^-$, one for each simple root determined by $\mathfrak{b}$. Then, $x = \sum e_i$ is a regular nilpotent element in $\mathfrak{g}$.*

7.15. Remark. This theorem can be proved using the fact that $\mathfrak{n}^{reg}$ is a single B-orbit consisting of regular nilpotent elements in $\mathfrak{g}$ since $n \in \mathfrak{n}^{reg}$. However, we will use the corollary above to prove it instead.

Proof of Theorem. Let $\alpha_1, \dots, \alpha_n$ be the set of simple roots determined by $\mathfrak{b}$ and let $\mathfrak{n}_{\alpha_1}, \dots, \mathfrak{n}_{\alpha_n}$ for the corresponding root spaces in $\mathfrak{n}$ (so that, in particular, $e_i \in \mathfrak{n}_{\alpha_i}$). Let $\mathfrak{n}_{-\alpha_1}, \dots, \mathfrak{n}_{-\alpha_n}$ denote the corresponding negative root spaces and $e_{-1}, \dots, e_{-n}$ the corresponding negative root vectors.

From the general structure theory of semisimple Lie algebras, there is, for each $1 \le i \le n$, a unique multiple h_i of $[e_i, e_{-i}]$ such that $\alpha_i(h_i) = 2$. Furthermore, the h_i's form a basis of $\mathfrak{h}$. Now, fix $h \in \mathfrak{h}$ so that $\alpha_i(h) = 2$ for all i and define complex numbers $\mu_1, \dots, \mu_n$ by the equation $h = \sum \mu_i h_i$. Let $y := \sum \mu_i e_{-i}$. Now, for any simple roots α, β where $\alpha \neq \beta$, $\alpha - \beta$ is not a root. Thus, $[e_i, e_{-j}] = 0$ whenever $i \neq j$ because . Now, we calculate

$$[h, x] = [h, \sum e_i] = \sum [h, e_i] = \sum_i \alpha_i(h) e_i = 2x$$

$$[h, y] = \sum_i \mu_i(-\alpha_i)(h) e_{-i} = -2y$$

$$[x, y] = \sum_{i,j} \mu_j [e_i, e_{-j}] = \sum_i \mu_i [e_i, e_{-i}] = \sum_i \mu_i h_i = h$$

Thus, (x, y, h) is an $\mathfrak{sl}_2$-triple and all the eigenvalues of h on $\mathfrak{g}$ are even since $\alpha_i(h) = 2$. By the corollary above, we obtain $\dim Z_{\mathfrak{g}}(x) = \dim Z_{\mathfrak{g}}(h) = \dim \mathfrak{h}$ and therefore x is regular. $\square$

why?

Where did this come from?

7.2. Consequences using Transversal Slices.

This whole section is completely confusing and I do not understand anything except Kostant's theorem.

7.16. Definition. ([**CG00**] 3.2.19) A locally closed (in the ordinary Hausdorff topology) complex analytic subset $S \subseteq X$ containing the point y will be called a *transverse slice* to Y at y if there is an open neighborhood of y (in the ordinary Hausdorff topology), $U \subseteq X$, and an analytic isomorphism $f\colon (Y \cap U) \times S \xrightarrow{\sim} U$ such that f restricts to the tautological maps of the factors

$$f\colon \{y\} \times S \xrightarrow{\sim} S \text{ and } (Y \cap U) \times \{y\} \xrightarrow{\sim} Y \cap U.$$

7.17. Lemma. *([**CG00**] 3.2.20) Let G be an algebraic group, V a smooth algebraic G-variety, X a G-stable algebraic subvariety in V, and S_V a locally-closed complex-analytic submanifold which is transverse to $\mathcal{O}$ at $y \in$*

$\mathcal{O}$. Then the intersection with X of a small enough open neighborhood of y in S_V is a transverse slice to $\mathcal{O}$ in X.

PROOF. See [**CG00**]. ☐

7.18. PROPOSITION. *([**CG00**] 3.7.15)Fix a nilpotent $e \in \mathfrak{g}$ and corresponding $\mathfrak{sl}_2$-triple (e, f, h). Let $\mathcal{O}$ be the G-conjugacy class of e. Then, the affice space $e + Z_{\mathfrak{g}}(f)$ is transverse to the orbit $\mathcal{O}$ in $\mathfrak{g}$. Moreover, we have $\mathcal{O} \cap (e + Z_{\mathfrak{g}}(f)) = e$.*

7.19. REMARK. The affine space $e + Z_{\mathfrak{g}}(f)$, or its intersection with $\mathcal{N}$, is often referred to as the *standard slice* to the orbit $\mathcal{O}$ at the point e. This is due to the fact that there exists a small enough neighborhood U containing e such that $\mathcal{N} \cap (e + Z_{\mathfrak{g}}(f)) \cap U$ is a transverse slice to $\mathcal{O}$ in $\mathcal{N}$.

7.20. COROLLARY. *([**CG00**] 3.7.19) The fiber $\mu^{-1}(e) \subseteq \tilde{\mathcal{N}}$ is a homotopy retract of the variety $\tilde{S} = \mu^{-1}(e + Z_{\mathfrak{g}}(f)) \subseteq \tilde{N}$.*

7.21. LEMMA. *([**CG00**] 3.7.21) Let $U \subseteq Z_G(e)$ be a unipotent normal subgroup corresponding to the Lie algebra $\mathfrak{u} = \bigoplus_{i>0} Z_{\mathfrak{g}_i}(e)$.*

(a) We have $\mathfrak{u} = \ker(e) \cap \operatorname{im}(e)$.
(b) The affine space $h + \mathfrak{u}$ is stable under the adjoint U-action; moreover, $h + \mathfrak{u} = U \cdot h$ is a single U-orbit.

PROOF OF KOSTANT'S THEOREM. Let (e, f, h) and (e, f', h') be two $\mathfrak{sl}_2$-triples. If $h = h'$, then

$$[f', e] = h' = h = [f, e] \Longrightarrow [f' - f, e] = 0 \text{ and } f' - f \in Z_{\mathfrak{g}}(e)$$

However, $f, f' \in \mathfrak{g}_{-2}$ in the grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ into ad_h-eigenspaces. Thus, $f' - f \in Z_{\mathfrak{g}_{-2}}(e) = 0$ and so $f' = f$.

Now, for any two $\mathfrak{sl}_2$-triples (e, f, h) and (e, f', h'), we get

$$[h', e] = [h, e] = 2e \Longrightarrow [h' - h, e] = 0 \Longrightarrow h' - h \in Z_{\mathfrak{g}}(e) = \ker(e)$$

Furthermore, we have

$$h' - h = [e, f' - f] \in \operatorname{im}(e) \Longrightarrow h' - h \in \ker(e) \cap \operatorname{im}(e)$$

So, by Lemma 7.21(a), we get $h' \in h + \mathfrak{u}$ and thus, by Lemma 7.21(b), there is a $u \in U$ such that $h' = uhu^{-1}$. Therefore,

$$e = u^{-1}eu, \ h = u^{-1}h'u, \ f'' := u^{-1}f'u$$

and so (e, h, f'') is an $\mathfrak{sl}_2$-triple for e. So, by above, $f'' = f$ and we are done. ☐

7.3. Correspondance between nilpotent orbits and G-conjugacy classes of $\mathfrak{sl}_2$-triples. This exposition is borrowed from chapter 3 of [**CM93**] and modified given what we already have proven.

7.22. Proposition. *For semisimple Lie algebra $\mathfrak{g}$, there is a bijection*

$$\mathrm{Hom}(\mathfrak{sl}_2, \mathfrak{g}) \setminus \{0\} \leftrightarrow \{\textit{standard triples in } \mathfrak{g}\}$$

Proof. Given a nonzero homomorphism $\phi\colon \mathfrak{sl}_2 \to \mathfrak{g}$, we get standard triple $\{\phi(e), \phi(f), \phi(h)\}$. Conversely, an $\mathfrak{sl}_2$-triple gives an injection of $\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$. □

7.23. Proposition. *There is a map defined by*

$$\Omega\colon \{G_{ad}\textit{-conjugacy classes of standard triples in } \mathfrak{g}\} \to \{\textit{nonzero nilpotent orbits in } \mathfrak{g}\}$$
$$\{e, f, h\} \mapsto \mathcal{O}_e$$

Proof. This map is induced by the fact that $\mathrm{Hom}(\mathfrak{sl}_2, \mathfrak{g}) \setminus \{0\}$ and standard triples in $\mathfrak{g}$ are invariant under the action of G_{ad}. □

7.24. Proposition. *The map Ω is surjective.*

Proof. By the Jacobson-Morozov theorem, every nilpotent $e \in \mathfrak{g}$ belongs to some $\mathfrak{sl}_2$-triple. Thus, there is a conjugacy class of triples $\{e, f, h\}$ mapping to $\mathcal{O}_e$. □

7.25. Proposition (Kostant's Theorem, rephrased). *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Any two standard triples (e, f, h) amd (e, f', h') with the same nilpositive element are conjugate by an element of G_{ad}, the adjoint group. Thus, the map*

$$\Omega\colon \{G_{ad}\textit{-conjugacy classes of standard triples in } \mathfrak{g}\} \to \{\textit{nonzero nilpotent orbits in } \mathfrak{g}\}$$

is injective.

7.26. Remark.

7.27. Theorem. *The map Ω is a one-to-one correspondance between the set of G_{ad}-conjugacy classes of standard triples in $\mathfrak{g}$ and the set of nonzero nilpotent orbits in $\mathfrak{g}$.*

7.28. Example. Consider $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Then, $G_{ad} = PSL_n$ and we can pick orbit representatives

$$e_{[d_1,\dots,d_k]} = \begin{pmatrix} J_{d_1}(0) & & & \\ & J_{d_2}(0) & & \\ & & \ddots & \\ & & & J_{d_k}(0) \end{pmatrix}$$

where $[d_1, \ldots, d_k]$ is a partition of n. As discussed in the $\mathfrak{sl}_n$ proof of the Jacobson-Morozov theorem, if we let

$$\rho_r(h) = \begin{pmatrix} r & & & & \\ & r-2 & & & \\ & & \ddots & & \\ & & & -r+2 & \\ & & & & -r \end{pmatrix} \quad \rho_r(f) = \begin{pmatrix} 0 & 0 & & & \\ \mu_1 & 0 & & & \\ & \ddots & \ddots & & \\ & & & 0 & 0 \\ & & & \mu_r & 0 \end{pmatrix}$$

where $\mu_i = i(r+1-i)$ for $1 \leq i \leq r$, we can associate $\mathfrak{sl}_2$-triple to $e_{[d_1,\ldots,d_k]}$ given by

$$h_{[d_1,\ldots,d_k]} = \rho_{d_1}(h) \oplus \cdots \oplus \rho_{d_k}(h), \;\; f_{[d_1,\ldots,d_k]} = \rho_{d_1}(f) \oplus \cdots \oplus \rho_{d_k}(f)$$

Conversely, given an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}_n$, $\{e, f, h\}$, the corresponding orbit is simply $\mathcal{O}_e$.

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