Flagged LLT polynomials and their applications

George H. Seelinger

ghseeli@umich.edu

joint work with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun

Center for Mathematics at the University of Notre Dame

Algebraic combinatorics and applications

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Symmetric polynomials and functions

- Symmetric polynomials $\mathbb{Q}[x_1, \ldots, x_n]^{S_n}$
- Generators

$$e_r(x_1, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} x_{i_1} \cdots x_{i_r}$$

$$\begin{aligned}
e_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3 \\
e_2(x_1, x_2, x_3) &= x_1x_2 + x_1x_3 + x_2x_3 \\
e_3(x_1, x_2, x_3) &= x_1x_2x_3
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- $\Lambda = \mathbb{Q}[e_1, e_2, \ldots]$
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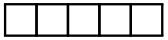
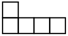
- Integer partitions of d.

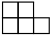
Partitions

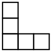
Definition

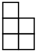
$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

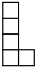
$5 \rightarrow$ 

$4 + 1 \rightarrow$ 

$3 + 2 \rightarrow$ 

$3 + 1 + 1 \rightarrow$ 

$2 + 2 + 1 \rightarrow$ 

$2 + 1 + 1 + 1 \rightarrow$ 

$1 + 1 + 1 + 1 + 1 \rightarrow$ 

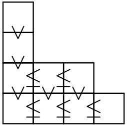
Young Tableaux

Definition

Filling of partition diagram of λ with numbers such that

1. strictly increasing up columns
2. weakly increasing along rows

Collection is called $\mathrm{SSYT}(\lambda)$.



For $\lambda = (2, 1)$,

$$\begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}$$

Schur polynomials

Associate a polynomial to SSYT(λ).

$$\begin{ytableau} 2 \\ 1 & 1 \end{ytableau}, \quad \begin{ytableau} 3 \\ 1 & 1 \end{ytableau}, \quad \begin{ytableau} 3 \\ 2 & 2 \end{ytableau}, \quad \begin{ytableau} 2 \\ 1 & 2 \end{ytableau}, \quad \begin{ytableau} 3 \\ 1 & 3 \end{ytableau}, \quad \begin{ytableau} 3 \\ 2 & 3 \end{ytableau}, \quad \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, \quad \begin{ytableau} 3 \\ 1 & 2 \end{ytableau}$$

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$$\to \begin{array}{|c|c}\hline x_2 & \\ \hline x_1 & \multicolumn{1}{c|}{x_1} \\ \hline\end{array}, \begin{array}{|c|c}\hline x_3 & \\ \hline x_1 & \multicolumn{1}{c|}{x_1} \\ \hline\end{array}, \begin{array}{|c|c}\hline x_3 & \\ \hline x_2 & \multicolumn{1}{c|}{x_2} \\ \hline\end{array}, \begin{array}{|c|c}\hline x_2 & \\ \hline x_1 & \multicolumn{1}{c|}{x_2} \\ \hline\end{array}, \begin{array}{|c|c}\hline x_3 & \\ \hline x_1 & \multicolumn{1}{c|}{x_3} \\ \hline\end{array}, \begin{array}{|c|c}\hline x_3 & \\ \hline x_2 & \multicolumn{1}{c|}{x_3} \\ \hline\end{array}, \begin{array}{|c|c}\hline x_2 & \\ \hline x_1 & \multicolumn{1}{c|}{x_3} \\ \hline\end{array}, \begin{array}{|c|c}\hline x_3 & \\ \hline x_1 & \multicolumn{1}{c|}{x_2} \\ \hline\end{array}$$

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- s_λ is a symmetric function.
- $\{s_\lambda\}_\lambda$ forms a basis for Λ.

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- $\mathrm{Frob}(\mathrm{Ind}_{S_1 \times S_1 \times S_1}^{S_3}(\mathbb{C} \times \mathbb{C} \times \mathbb{C})) = (s_{\ydiagram{1}})^3 = s_{\ydiagram{3}} + 2 s_{\ydiagram{1,2}} + s_{\ydiagram{1,1,1}}$

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Hidden Guide: Schur Positivity

"Naturally occurring" symmetric functions which are non-negative (coefficients in $\mathbb{N}$) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

Products of Schur polynomials

- Littlewood-Richardson rule: $s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$ for $c^\nu_{\lambda\mu} \in \mathbb{N}$.

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- Littlewood-Richardson rule: $s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$ for $c^\nu_{\lambda\mu} \in \mathbb{N}$.
- Skew Schur function: $s_{\nu/\lambda} = \sum_{T \in \mathsf{SSYT}(\nu/\lambda)} x^T = \sum_\mu c^\nu_{\lambda\mu} s_\mu$.

Products of Schur polynomials

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$$\begin{ytableau} 1 & 2 \\ \none & 1 & 1 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ \none & 1 & 2 \end{ytableau} \quad \begin{ytableau} 2 & 2 \\ \none & 1 & 1 \end{ytableau} \quad \cdots$$

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$$\begin{array}{|c|c|c}\hline 1&2&\\\hline &1&\multicolumn{1}{c|}{1}\\\cline{2-3}\end{array} \quad \begin{array}{|c|c|c}\hline 1&2&\\\hline &1&\multicolumn{1}{c|}{2}\\\cline{2-3}\end{array} \quad \begin{array}{|c|c|c}\hline 2&2&\\\hline &1&\multicolumn{1}{c|}{1}\\\cline{2-3}\end{array} \quad \cdots$$

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- q-deformation?

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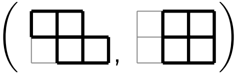
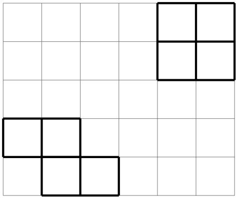
$\boldsymbol{\nu} =$ 



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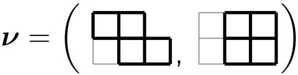

-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

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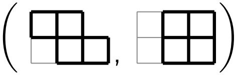
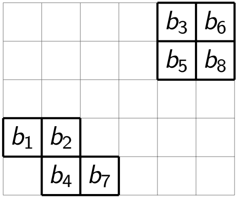
$\boldsymbol{\nu} =$ 



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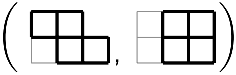
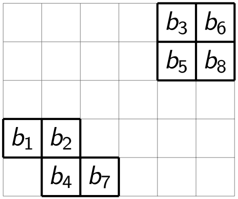
$\boldsymbol{\nu} =$ 



Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

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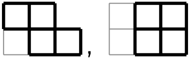
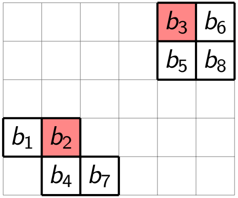
$\boldsymbol{\nu} = \Bigg($  $\Bigg)$



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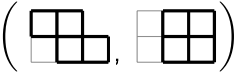
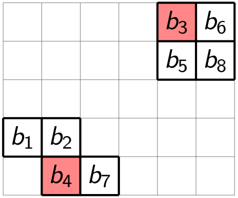
$\boldsymbol{\nu} =$ 



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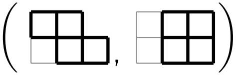
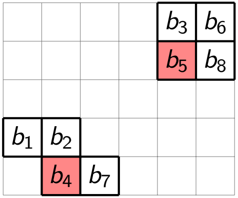
$\boldsymbol{\nu} =$ 



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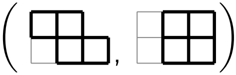
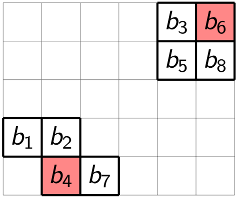
$\boldsymbol{\nu} =$ 



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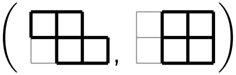
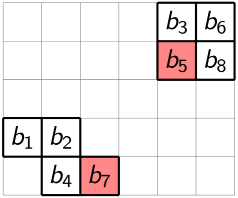
$\boldsymbol{\nu} =$ 



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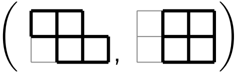
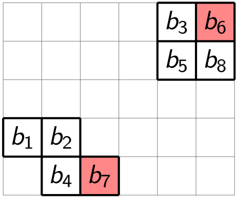
$\boldsymbol{\nu} =$ 



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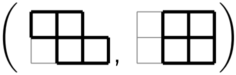
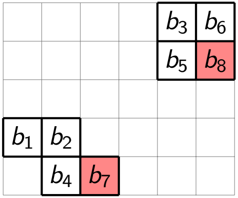
$\boldsymbol{\nu} =$ 



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LLT Polynomials

- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

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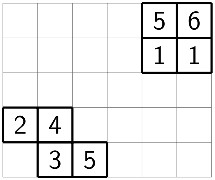
$$x^T = \textstyle\prod_{a \in \boldsymbol{\nu}} x_{T(a)}.$$

$T \quad =$



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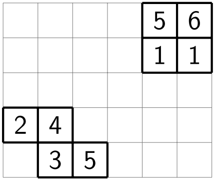

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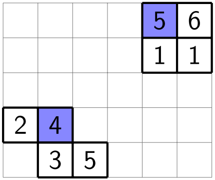

non-inversion

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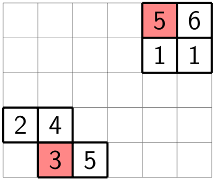

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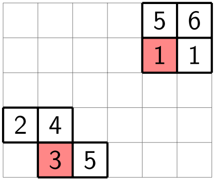
$T \quad =$ 

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LLT Polynomials

- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
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$$\mathcal{G}_{\boldsymbol{\nu}}(x; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} x^T,$$

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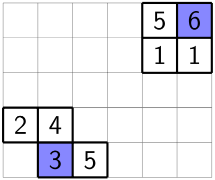

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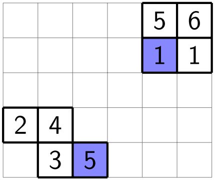

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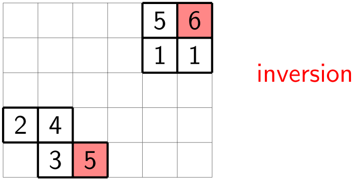

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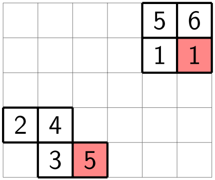

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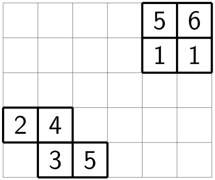

$\mathrm{inv}(T) = 4, \quad x^T = x_1^2 x_2 x_3 x_4 x_5^2 x_6$

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- Thus, $\mathcal{G}_{\boldsymbol{\nu}}$ are useful to show combinatorial quantities are Schur-positive!

Some notable occurrences of LLT Polynomials

- Haglund-Haiman-Loehr formula for Macdonald polynomials:

$$\tilde{H}_\mu(x;q,t) = t^{n(\mu)} \sum_R \Big(\prod_{\substack{\boxed{u} \\ \boxed{}}} q^{a+1} t^{l} \Big) \mathcal{G}_R(x;t^{-1})\,.$$

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- For $G =$ incomparability graph for natural unit interval order (encoded by Dyck path P), the t-chromatic symmetric polynomial is essentially a "signed" unicellular LLT polynomial (later), i.e.,

$$\chi_G(x;t) = (1-t)^{-|V|} \mathcal{G}_{\nu(P)}[(1-t)x;t]\,.$$

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- A *flagged semistandard Young tableau* of shape λ with flag $\mathbf{b}$ is a $T \in \mathrm{SSYT}(\lambda)$ such that the entries of row i are bounded above by b_i.
- Denote the set of such tableaux via $\mathrm{FT}(\lambda, \mathbf{b})$.
- E.g., $\lambda = (2,1), \mathbf{b} = (1,3)$, $\mathrm{FT}(\lambda, \mathbf{b}) =$

$$\begin{array}{|c|cl} \hline 2 & \multicolumn{2}{|l}{\scriptstyle \leq 3} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \scriptstyle \leq 1 \\ \hline \end{array} \qquad \begin{array}{|c|cl} \hline 3 & \multicolumn{2}{|l}{\scriptstyle \leq 3} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \scriptstyle \leq 1 \\ \hline \end{array}$$

Flagged Schur functions (Lascoux-Schützenberger, Wachs)

- For partition λ and flag $\mathbf{b}$, the *flagged Schur function* is given by

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- Schubert $\mathfrak{S}_w(x) = s_{\lambda,\mathbf{b}}(x)$ for *vexillary* $w \in S_n$ (2143-avoiding).
- $s_{\lambda,\mathbf{b}}(x)$ are examples of *Demazure characters* (Reiner-Shimozono, 1995).

Demazure characters

The *Demazure operator* π_i acts on $f \in \Bbbk[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$ by

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The *Demazure characters* or *key polynomials* are constructed from

- $\mathcal{D}_\lambda = x^\lambda := x_1^{\lambda_1} \cdots x_N^{\lambda_N}$ for partition λ.
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$$\begin{aligned}
\mathcal{D}_{520} &= x_1^5 x_2^2 \\
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\mathcal{D}_{205} &= \pi_2 \mathcal{D}_{250} = \pi_2\left(x_1^5 x_2^2 + x_1^4 x_2^3 + x_1^3 x_2^4 + x_1^2 x_2^5\right)
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- Demazure characters are characters of irreducible B-modules.

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Demazure atoms are defined the same as keys but with $\hat{\pi}_i := \pi_i - 1$ in place of π_i:

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- Demazure atoms are related to Demazure characters by Bruhat order inclusion-exclusion:

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\mathcal{D}_{520} &= \mathcal{A}_{520} \\
\mathcal{D}_{250} &= \mathcal{A}_{520} + \mathcal{A}_{250} \\
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- Demazure atoms form a dual basis of $\mathbb{k}[x_1, \ldots, x_N]$ to Demazure characters under a nice inner product.

Recovering Symmetric Functions

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$$\pi_{w_0}(\mathcal{A}_\alpha) = \begin{cases} s_\alpha & \alpha \text{ a partition,} \\ 0 & \text{else.} \end{cases}$$

Summary of families

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\begin{tikzcd}
\mathcal{A}_\alpha \arrow[rr, dash, dashed, "\mathcal{D} = \text{ positive sum of } \mathcal{A}"] \arrow[d] \arrow[drr] & & \mathcal{D}_\alpha \arrow[rr, dash, dashed, "s_{\lambda,\mathbf{b}} = \text{ special case of } \mathcal{D}"] \arrow[d] & & s_{\lambda,\mathbf{b}} \arrow[dll, "\text{Weyl sym. } \pi_{w_0}"'] \\
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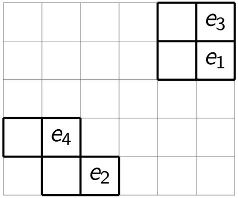
Question

Is there some "nice" family of polynomials that Weyl symmetrizes to LLT polynomials?

Flagged LLT Polynomials (Blasiak-Haiman-Morse-Pun-S.)

- Let $e_1, \ldots, e_l$ be the row ends of $\boldsymbol{\nu}$, ordered in reverse reading order.
- Fix flag $\mathbf{b} = (b_1 \le \cdots \le b_l)$.
- $\mathrm{FT}(\boldsymbol{\nu}, \mathbf{b})$ = set of semistandard tableaux T on $\boldsymbol{\nu}$ satisfying $T(e_i) \le b_i$.
- The *flagged LLT polynomial* indexed by $\boldsymbol{\nu}$ and $\mathbf{b}$ is

$$\mathcal{G}_{\boldsymbol{\nu},\mathbf{b}}(x;t) = \sum_{T \in \mathrm{FT}(\boldsymbol{\nu},\mathbf{b})} t^{\mathrm{inv}(T)} x^T .$$



Flagged LLT Polynomials (Blasiak-Haiman-Morse-Pun-S.)

- Let $e_1, \ldots, e_l$ be the row ends of $\boldsymbol{\nu}$, ordered in reverse reading order.
- Fix flag $\mathbf{b} = (b_1 \leq \cdots \leq b_l)$.
- $\mathrm{FT}(\boldsymbol{\nu}, \mathbf{b})$ = set of semistandard tableaux T on $\boldsymbol{\nu}$ satisfying $T(e_i) \leq b_i$.
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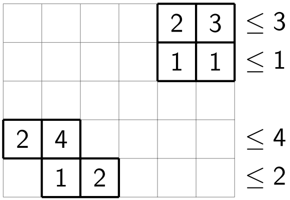
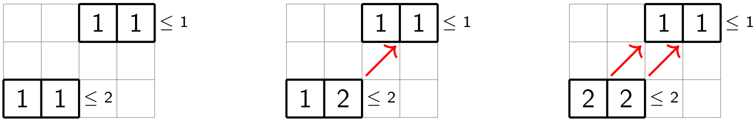


Flagged LLT Polynomials

$$\boldsymbol{\nu} = \left(\square\square, \square\square\right), \mathbf{b} = (1,2)$$

T



$$\mathcal{G}_{r,\boldsymbol{\nu}}(x;t) = x_1^4 + t\, x_1^3 x_2 + t^2 x_1^2 x_2^2$$

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- $\mathcal{G}_{\boldsymbol{\nu},\mathbf{b}}(x;t)$ are conjecturally positive in terms of Demazure atoms.

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The *signed flagged LLT polynomial* indexed by $\boldsymbol{\nu}$ and flag $\mathbf{b}$ is

$$\mathcal{G}^{\pm}_{\boldsymbol{\nu},\mathbf{b}}(x;t) = \sum_{T \in \mathsf{FT}^{\pm}(\boldsymbol{\nu},\mathbf{b})} t^{\mathsf{inv}(T)}(-t)^{-\#\mathsf{bar}(T)} x^{|T|},$$

where $|T|$ is the result of removing all bars from T.

Signed flagged LLT polynomials

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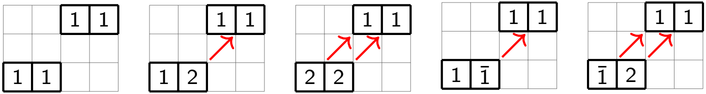
T 

$$\begin{aligned} \mathcal{G}^{\pm}_{\nu,\mathbf{b}}(x;t) &= x_1^4 + tx_1^3x_2 + t^2x_1^2x_2^2 - x_1^4 - tx_1^3x_2 \\ &= t^2x_1^2x_2^2 \end{aligned}$$

Remark

Signed flagged LLTs are typically not positive in terms of Demazure atoms, nor are they generally even monomial positive.

Flagged plethysm

Define *flagged plethysm* $\Pi_{t,x} \colon \Bbbk[x_1, \ldots, x_l] \to \Bbbk[x_1, \ldots, x_l]$ to be the (over-determined) linear map

$$\Pi_{t,x} \mathcal{G}^{\pm}_{\boldsymbol{\nu},\mathbf{b}}(x; t^{-1}) = \mathcal{G}_{\boldsymbol{\nu},\mathbf{b}}(x; t^{-1})\,.$$

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$\Pi_{t,x}$ *is well-defined.*

$\Pi_{t,x}$ is a "nonsymmetric analogue" of the plethystic map $f[X] \mapsto f[X/(1-t)]$ for symmetric function f.

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2. A "nonsymmetric shuffle theorem" expressed in terms of flagged LLT polynomials associated to flagged Dyck paths.
3. Tewari-Wilson-Zhang define chromatic nonsymmetric polynomials associated to $d \times d$ Dyck paths starting with r north steps, $\chi_{\mathbf{b},\pi}$. These are (essentially) examples of signed flagged LLT polynomials.

At this point, I decided to jump to slide titled “Raising the flag.”

Macdonald polynomials

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Modified Macdonald polynomials

- Plethystically modified Macdonald polynomials
$$H_\mu(x; q, t) = J_\mu[x/(1-t); q, t].$$
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$$H_{31} = ts_4 + (1+qt+q^2t)s_{31} + (q+tq^2)s_{22} + (q+q^2+q^3t)s_{211} + q^3s_{1111}$$

Haglund-Haiman-Loehr formula example

$$H_\mu(X;q,t) = t^{n(\mu)} \sum_R \big(\prod_{\begin{array}{|c|}\hline u \\ \hline \\ \hline \end{array}} q^{a+1} t^{l} \big) \mathcal{G}_R(X;t^{-1})$$

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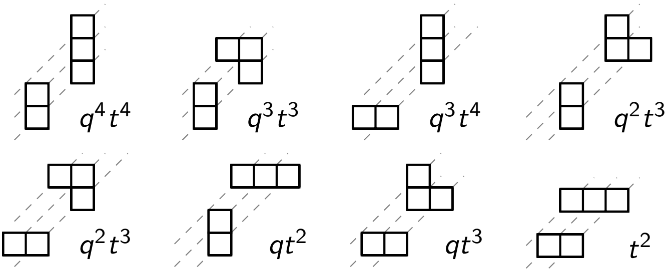
For $\mu = (3,2)$, we get 8 LLT terms:



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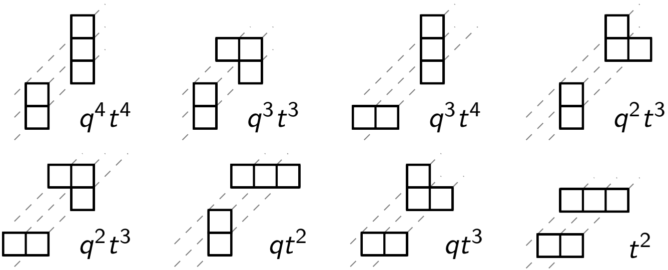

Remark: Can permute the order of ribbons (but changes arm and leg statistics and LLTs.)

Nonsymmetric Macdonald polynomials

The Cherednik operators $Y_1, \ldots, Y_N$ act on $\mathbb{Q}(q,t)[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$.

$$T_i f = s_i f + (1-t) x_i \, \frac{f - s_i f}{x_i - x_{i+1}}\,, \qquad \text{(Demazure-Lusztig operators)}$$

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- The $\mathcal{E}_\alpha$'s "Hecke symmetrize" to the integral form J_μ's.

Nonsymmetric HHL formula example

$$\mathcal{E}_\alpha(x;q,t) = t^{n(\mu_+)} \sum_R \Big(\prod_{\substack{\boxed{u} \\ \boxed{}}} q^{a+1} t^{l} \Big) \mathcal{G}^{\pm}_{R,(1,2,\ldots,m)}(x;t^{-1}).$$

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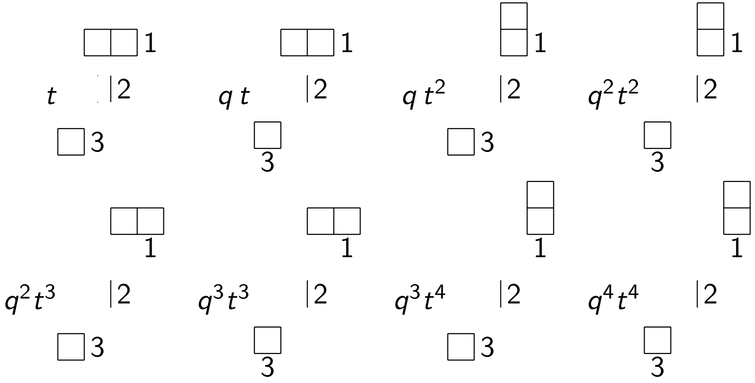

- Convention: bound boxes are always on the same content "diagonal."
- Variation: bound boxes below shape bound shape strictly from below.

The Missing Corner

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{E}_\alpha(x_1, \ldots, x_N; q, t) \arrow[rr] \arrow[d, "\text{Hecke sym.}"'] & & \textbf{?} \arrow[d, "\text{Weyl sym.}"] \\
J_\mu(x; q, t) \arrow[rr, "\text{Plethysm}"'] & & H_\mu(x; q, t)
\end{tikzcd}$$

Question

Is there a "nice" family of polynomials that Weyl symmetrizes to the modified Macdonald polynomials?

An important feature for "nice"? Positivity!

Toward positivity

- Knop (2007) formulated a positivity conjecture for a stable version of $\mathcal{E}_\alpha$ involving Kazhdan-Lusztig theory.
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- We can understand stabilizations in terms of signed flagged LLT combinatorics.
- Upshot: look at stabilizations for "positivity."

Filling in the missing corner

$$\begin{tikzcd}
\begin{array}{c}\mathcal{E}_\alpha(x_1,\ldots,x_N;q,t) = \\ \sum \text{signed flagged LLTs}\end{array} \arrow[d, rightsquigarrow, "\text{stabilize}"] & \\
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$$

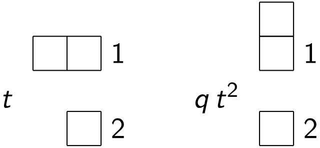
Modern Macdonald polynomials

Definition (Blasiak-Haiman-Morse-Pun-S., 2025+)

The *modern Macdonald polynomial* indexed by $\eta \in \mathbb{N}^r$ and partition λ is

$$\mathrm{nsH}_{\eta|\lambda}(x;q,t) = t^{n((\eta;\lambda)_+)} \sum_{R} \Big(\prod_{\substack{\boxed{u} \\ \boxed{}}} q^{a+1} t^{l} \Big) \mathcal{G}_{R,(1,2,\ldots,r)}(x;t^{-1})$$

For $r = 2$, $\eta = (2,1)$, $\lambda = \varnothing$, the flagged LLT terms are



$$\mathrm{nsH}_{21|\varnothing}(x_1,x_2,x_3;q,t) = t\,x_1^3 + x_1^2x_2 + qt\,x_1^2x_2 + qt\,x_1x_2^2 + qt\,x_1^2x_3 + q\,x_1x_2x_3$$

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- $\mathrm{nsH}_{\eta|\lambda}(x;q,t)$ is monomial positive with a combinatorial formula.
- $\mathrm{nsH}_{\eta|\lambda}(x;q,t)$ Weyl symmetrizes to symmetric $H_{\beta_+}(x;q,t)$.
- We conjecture $\mathrm{nsH}_{\eta|\lambda}(x;q,t)$ is atom positive (would follow from flagged LLTs being atom positive).

Raising the flag

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Thank you!

Filling in the missing corner

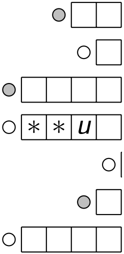
- $\mathcal{P}(r) = \mathbb{Q}(q,t)[x_1,\ldots,x_r] \otimes \Lambda_{\mathbb{Q}(q,t)}(x_{r+1},\ldots)$.
- Use a "stable" version of $\mathcal{E}_\alpha$, which we denote $\mathcal{J}_{\eta|\lambda}$.

$$\begin{tikzcd}
\mathcal{J}_{\eta|\lambda}(x;q,t) \arrow[r, "\Pi_r"] \arrow[d, "\text{Hecke sym.}"'] & \mathrm{nsH}_{\eta|\lambda}(x;q,t) \arrow[d, "\text{Weyl sym.}"] \\
J_{(\eta;\lambda)_+}(x;q,t) \arrow[r, "\text{Plethysm}"'] & H_{(\eta;\lambda)_+}(x;q,t)
\end{tikzcd}$$

- $\mathcal{J}_{\eta|\lambda}(x;q,t)$ and $\mathrm{nsH}_{\eta|\lambda}(x;q,t)$ for $(\eta|\lambda) \in \mathbb{N}^r \times \mathrm{Par}$ form bases of $\mathcal{P}(r)$.
- $(\eta;\lambda)_+ =$ partition rearrangement of concatenating $(\eta;\lambda)$.
- Π_r is "r-nonsymmetric plethysm"

$$\Pi_r(f(x_1,\ldots,x_r) \otimes g(x)) = (\Pi_{t,x} f(x_1,\ldots,x_r)) \otimes g\left[x/(1-t)\right].$$

Arms and legs



$$a(u) = 2, l(u) = 3$$