K-theoretic Catalan functions

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Overview

1. Schubert calculus
2. Catalan functions
3. K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

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Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c^{\nu}_{\lambda\mu} f_\nu$

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Representatives

Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c^{\nu}_{\lambda\mu} s_\nu$ for combinatorially understood Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$.

Schur polynomials and raising operators

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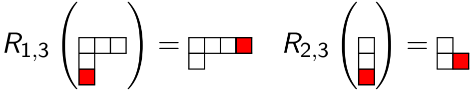

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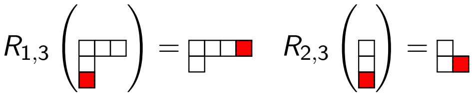

- Schur function $s_\lambda = \prod_{i<j}(1 - R_{ij}) h_\lambda$ (Jacobi-Trudi)

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Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur-P and Q functions
(Co)homology of affine Grassmannian	(dual) k-Schur functions
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K-homology of affine Grassmannian	K-k-Schur functions

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K-theory and K-homology of the affine Grassmannian

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Simulatenously generalizes K-theory of Grassmannian and (co)homology of affine Grassmannian.

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2. Homology classes of affine Grassmannian represented by k-Schur functions ($t = 1$).
3. (Lam et al., 2010) leave open the question: what is a direct formulation of the K-homology representatives of the affine Grassmannian (K-k-Schur functions)?

Goal

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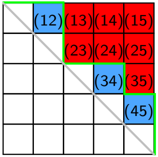
Identify K-k-Schur functions in explicit (simple) terms amenable to calculation and proofs.

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



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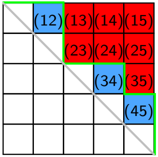

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Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^\ell$

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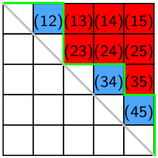

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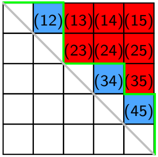

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- $\Psi = \text{all roots} \implies H(\Psi;\gamma) = h_\gamma$

Catalan functions

k-Schur root ideal for λ

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$$\begin{aligned}\Psi = \Delta^k(\lambda) &= \{(i,j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

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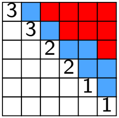
$\Delta^4(3,3,2,2,1,1) =$  $\leftarrow$ row i has $4 - \lambda_i$ non-roots

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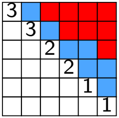

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k-Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda)\,.$$

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Remark

(Blasiak et al., 2019) show results for k-Schur functions with parameter t, but $t = 1$ specialization is necessary for Schubert calculus.

Lowering Operators

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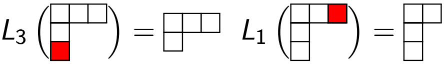
- Extra ingredient: lowering operators $L_j(h_\lambda) = h_{\lambda-\epsilon_j}$



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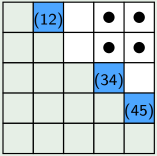
Example

non-roots of Ψ in blue, roots of $\mathcal{L}$ marked with $\bullet$



$$K(\Psi;\mathcal{L};54332)$$
$$= (1 - L_4)^2(1 - L_5)^2$$
$$\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$$

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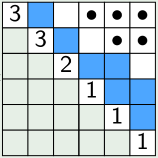
Example

$g^{(4)}_{332111} =$ 

$\Delta^+ \setminus \Psi = \Delta^+_6 \setminus \Delta^{(4)}(332111), \mathcal{L} = \Delta^{(5)}(332111)$

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Theorem (Blasiak-Morse-S., 2022)

The $g_\lambda^{(k)}$ *"branching coefficients" are alternating by degree, i.e. the* $b_{\lambda\mu}^{(k)}$ *in*

$$g_\lambda^{(k)} = \sum_\mu b_{\lambda\mu}^{(k)} g_\mu^{(k+1)}$$

satisfy $(-1)^{|\lambda|-|\mu|} b_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$.

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Under the Peterson Isomorphism, the "quantum Grothendieck polynomials" $\mathfrak{G}_w(z; Q)$ get sent to "closed K-k-Schur functions", $\mathfrak{g}_\lambda^{(k)} = K(\Delta^{(k)}; \Delta^{(k)}; \lambda)$ with suitable localization.

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Proved using "Katalan function description."

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3. Combinatorially describe $g_\lambda^{(k)} = \sum_\mu ??s_\mu^{(k)}$.
4. Answer same questions for "closed K-k-Schur's."

Other results using Catalan function methods

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3. New formulas for Macdonald polynomials using raising operators (Blasiak-Haiman-Morse-Pun-S.)

References

Thank you!

Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. 2023. *A Shuffle Theorem for Paths Under Any Line*, Forum of Mathematics, Pi **11**.

________. 2021. *Dens, Nests and the Loehr-Warrington Conjecture*, arXiv.

Blasiak, Jonah, Jennifer Morse, and Anna Pun. 2020. *Demazure Crystals and the Schur Positivity of Catalan Functions*, arXiv.

Blasiak, Jonah, Jennifer Morse, Anna Pun, and Daniel Summers. 2019. *Catalan Functions and k-Schur Positivity*, Journal of the AMS.

Blasiak, Jonah, Jennifer Morse, and George H. Seelinger. 2022. *K-Theoretic Catalan Functions*, Advances in Mathematics **404**.

Chen, Li-Chung. 2010. *Skew-linked partitions and a representation theoretic model for k-Schur functions*, Ph.D. thesis.

References

Thank you!

Ikeda, Takeshi, Shinsuke Iwao, and Toshiaki Maeno. 2020. *Peterson Isomorphism in K-theory and Relativistic Toda Lattice*, International Mathematics Research Notices **19**.

Ikeda, Takeshi, Shinsuke Iwao, and Satoshi Naito. 2022. *Closed k-Schur Katalan Functions as K-Homology Schubert Representatives of the Affine Grassmannian*, arXiv.

Lam, Thomas, Anne Schilling, and Mark Shimozono. 2010. *K-theory Schubert calculus of the affine Grassmannian*, Compositio Math. **146**, 811–852.

Morse, Jennifer. 2011. *Combinatorics of the K-theory of affine Grassmannians*, Advances in Mathematics.

Panyushev, Dmitri I. 2010. *Generalised Kostka-Foulkes polynomials and cohomology of line bundles on homogeneous vector bundles*, Selecta Math. (N.S.) **16**, no. 2, 315–342.