Flagged LLT polynomials and nonsymmetric Macdonald polynomials

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UC Davis Algebra & Discrete Mathematics Seminar

joint work with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun

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Symmetric polynomials and functions

- Symmetric polynomials $\mathbb{Q}[x_1, \ldots, x_n]^{S_n}$
- Generators

$$e_r(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} x_{i_1} \cdots x_{i_r}$$

$$\begin{aligned}
e_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3 \\
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- Integer partitions of d.

Partitions

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

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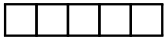
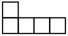
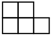
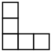
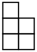
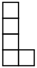
$5 \rightarrow$ 

$4 + 1 \rightarrow$ 

$3 + 2 \rightarrow$ 

$3 + 1 + 1 \rightarrow$ 

$2 + 2 + 1 \rightarrow$ 

$2 + 1 + 1 + 1 \rightarrow$ 

$1 + 1 + 1 + 1 + 1 \rightarrow$ 

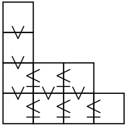
Young Tableaux

Definition

Filling of partition diagram of λ with numbers such that

1. strictly increasing up columns
2. weakly increasing along rows

Collection is called SSYT(λ).



For $\lambda = (2, 1)$,

$$\begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline\end{array}$$

Schur polynomials

Associate a polynomial to SSYT(λ).

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- $\{s_\lambda\}_\lambda$ forms a basis for Λ.

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Hidden Guide: Schur Positivity

"Naturally occurring" symmetric functions which are non-negative (coefficients in $\mathbb{N}$) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

Products of Schur polynomials

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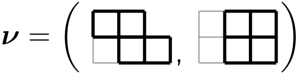
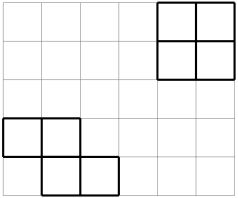
- q-deformation?

Key Object: LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)





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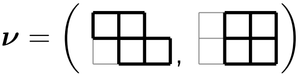

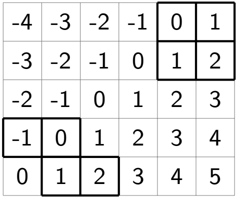

-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

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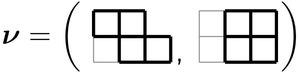

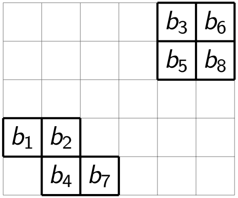

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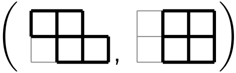
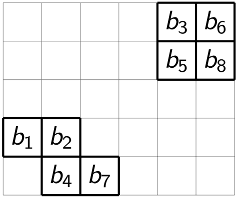
$\boldsymbol{\nu} =$ 



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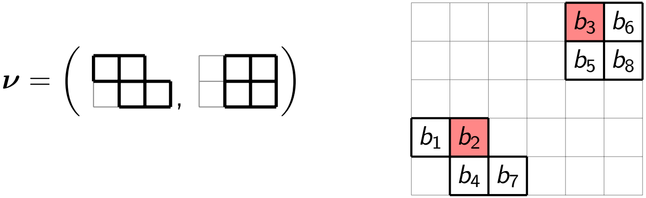

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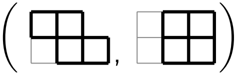

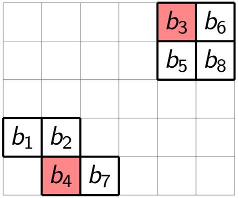

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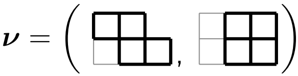

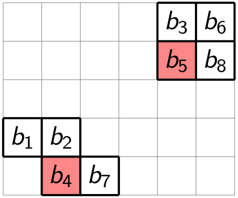

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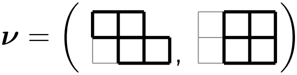

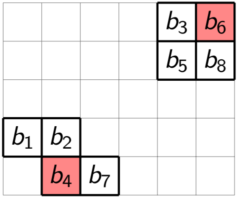

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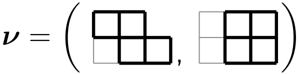

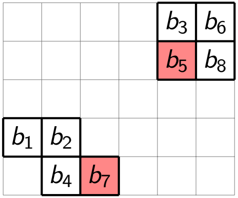

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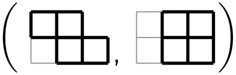

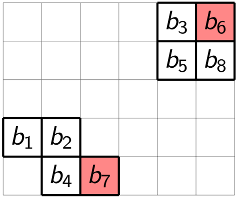

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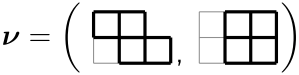

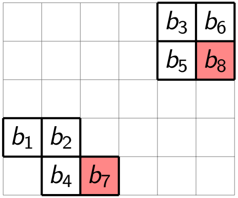

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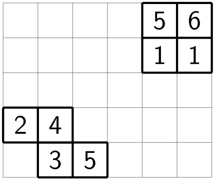

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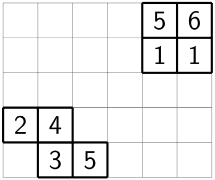

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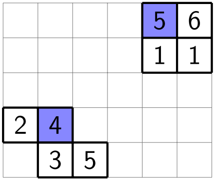

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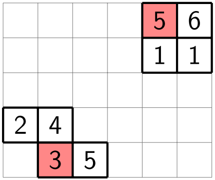

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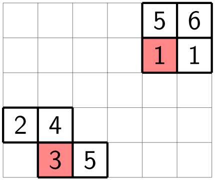

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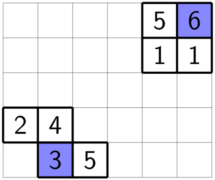

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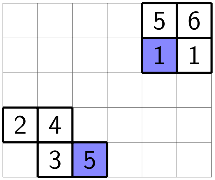

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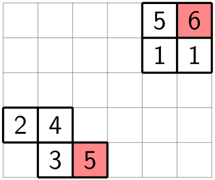

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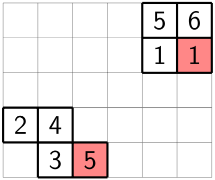

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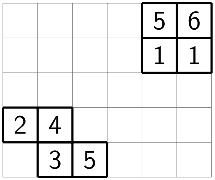

$\mathrm{inv}(T) = 4, \quad x^T = x_1^2 x_2 x_3 x_4 x_5^2 x_6$

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- $\mathcal{G}_{\boldsymbol{\nu}}$ is Schur-positive for any tuple of skew shapes $\boldsymbol{\nu}$ [Grojnowski-Haiman, 2007].
- Thus, $\mathcal{G}_{\boldsymbol{\nu}}$ are useful to show combinatorial quantities are Schur-positive!

Some notable occurrences of LLT Polynomials

- Haglund-Haiman-Loehr formula for Macdonald polynomials:

$$\tilde{H}_\mu(x;q,t) = t^{n(\mu)} \sum_R \Big(\prod_{\begin{array}{|c|}\hline u \\ \hline \\ \hline \end{array}} q^{a+1} t^{l} \Big) \mathcal{G}_R(x;t^{-1})\,.$$

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- For G = incomparability graph for natural unit interval order (encoded by Dyck path P), the t-chromatic symmetric polynomial is a "signed" LLT polynomial (later), i.e.,

$$\chi_G(x;t) = (1-t)^{-|V|} \mathcal{G}_{\nu(P)}[(1-t)x;t]\,.$$

Flagged Tableaux

- Fix partition shape $\lambda = (\lambda_1, \dots, \lambda_l)$ and *flag* $\mathbf{b} = (b_1 \leq \dots \leq b_l) \in \mathbb{N}^l$.

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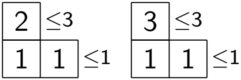
- A *flagged semistandard Young tableau* of shape λ with flag $\mathbf{b}$ is a $T \in \mathrm{SSYT}(\lambda)$ such that the entries of row i are bounded above by b_i.
- Denote the set of such tableaux via $\mathrm{FT}(\lambda, \mathbf{b})$.
- E.g., $\lambda = (2, 1), \mathbf{b} = (1, 3)$, $\mathrm{FT}(\lambda, \mathbf{b}) =$



Flagged Schur functions (Lascoux-Schützenberger, Wachs)

- For partition λ and flag $\mathbf{b}$, the *flagged Schur function* is given by

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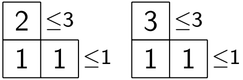

$$\begin{pmatrix} h_2(x_1) & h_3(x_1) \\ h_0(x_1, x_2, x_3) & h_1(x_1, x_2, x_3) \end{pmatrix}$$

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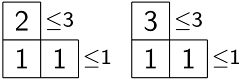

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$$\begin{array}{|c|c|l}\cline{1-1} 2 & \multicolumn{2}{l}{\leq 3} \\ \cline{1-2} 1 & 1 & \leq 1 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|l}\cline{1-1} 3 & \multicolumn{2}{l}{\leq 3} \\ \cline{1-2} 1 & 1 & \leq 1 \\ \cline{1-2}\end{array} \quad \begin{pmatrix} h_2(x_1) & h_3(x_1) \\ h_0(x_1, x_2, x_3) & h_1(x_1, x_2, x_3) \end{pmatrix}$$

- Schubert $\mathfrak{S}_w(x) = s_{\lambda,\mathbf{b}}(x)$ for *vexillary* $w \in S_n$ (2143-avoiding).
- $s_{\lambda,\mathbf{b}}(x)$ are examples of *Demazure characters*.

Demazure characters and atoms

The *Demazure operator* π_i acts on $f \in \mathbb{Q}(q,t)[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$ by

$$\pi_i(f) = \frac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}.$$

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The *Demazure characters* or *key polynomials* are constructed from

- $\mathcal{D}_\lambda = x^\lambda := x_1^{\lambda_1} \cdots x_N^{\lambda_N}$ for partition λ.
- $\mathcal{D}_{s_i(\alpha)} = \pi_i \mathcal{D}_\alpha$ for $\alpha_i > \alpha_{i+1}$, for any $\alpha \in \mathbb{N}^N$.

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Demazure atoms are defined the same as keys but with $\hat{\pi}_i := \pi_i - 1$ in place of π_i:

- $\mathcal{A}_\lambda = x^\lambda$ for partition λ.
- $\mathcal{A}_{s_i(\alpha)} = \hat{\pi}_i \mathcal{A}_\alpha$ for $\alpha_i > \alpha_{i+1}$, for any $\alpha \in \mathbb{N}^N$.

Examples

$$\mathcal{D}_{520} = x_1^5 x_2^2$$

$$\mathcal{D}_{250} = \pi_1 \mathcal{D}_{520} = \pi_1(x_1^5 x_2^2) = x_1^5 x_2^2 + x_1^4 x_2^3 + x_1^3 x_2^4 + x_1^2 x_2^5$$

$$\mathcal{D}_{205} = \pi_2 \mathcal{D}_{250} = \pi_2\left(x_1^5 x_2^2 + x_1^4 x_2^3 + x_1^3 x_2^4 + x_1^2 x_2^5\right)$$

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\begin{aligned}
\mathcal{D}_{520} &= \mathcal{A}_{520} \\
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\mathcal{D}_{205} &= \mathcal{A}_{520} + \mathcal{A}_{250} + \mathcal{A}_{502} + \mathcal{A}_{205} \\
&\vdots
\end{aligned}
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Recovering Symmetric Functions

- If $\mathbf{b} = (n, \ldots, n)$ for $n = |\lambda|$, then $s_{\lambda,\mathbf{b}}(x) = s_\lambda(x_1, \ldots, x_n)$.

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- For *Weyl symmetrization operator* π_{w_0}, we have

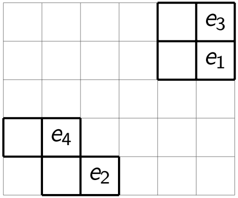
$$\pi_{w_0}(s_{\lambda,\mathbf{b}}(x)) = s_\lambda(x), \quad \pi_{w_0}(\mathcal{D}_\alpha) = \mathcal{D}_{\mathrm{sort}(\alpha)} = s_{\alpha_+},$$

$$\pi_{w_0}(\mathcal{A}_\alpha) = \begin{cases} s_\alpha & \alpha \text{ a partition,} \\ 0 & \text{else.} \end{cases}$$

Flagged LLT Polynomials (Blasiak-Haiman-Morse-Pun-S.)

- Let $e_1, \ldots, e_l$ be the row ends of $\boldsymbol{\nu}$, ordered in reverse reading order.
- Fix flag $\mathbf{b} = (b_1 \leq \cdots \leq b_l)$.
- $\mathrm{FT}(\boldsymbol{\nu}, \mathbf{b})$ = set of semistandard tableaux T on $\boldsymbol{\nu}$ satisfying $T(e_i) \leq b_i$.
- The *flagged LLT polynomial* indexed by $\boldsymbol{\nu}$ and $\mathbf{b}$ is

$$\mathcal{G}_{\boldsymbol{\nu},\mathbf{b}}(x;t) = \sum_{T \in \mathrm{FT}(\boldsymbol{\nu},\mathbf{b})} t^{\mathrm{inv}(T)} x^T .$$



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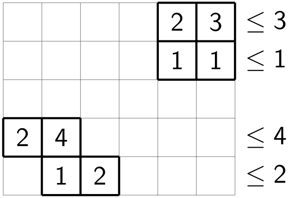
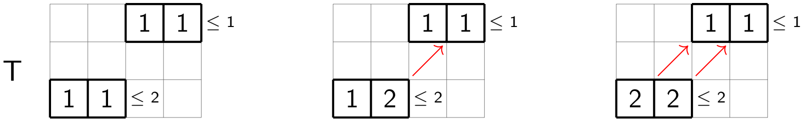


Flagged LLT Polynomials

$$\boldsymbol{\nu} = \left(\square\square, \square\square\right), \mathbf{b} = (1, 2)$$



$$\mathcal{G}_{r,\boldsymbol{\nu}}(x;t) = x_1^4 + t\, x_1^3 x_2 + t^2 x_1^2 x_2^2$$

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- $\mathcal{G}_{\boldsymbol{\nu},\mathbf{b}}(x;t)$ are conjecturally positive in terms of *Demazure atoms*.

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The *signed flagged LLT polynomial* indexed by $\boldsymbol{\nu}$ and flag $\mathbf{b}$ is

$$\mathcal{G}^{\pm}_{\boldsymbol{\nu},\mathbf{b}}(x;t) = \sum_{T \in \mathsf{FT}^{\pm}(\boldsymbol{\nu},\mathbf{b})} t^{\mathrm{inv}(T)}(-t)^{-\#\mathrm{bar}(T)} x^{|T|},$$

where $|T|$ is the result of removing all bars from T.

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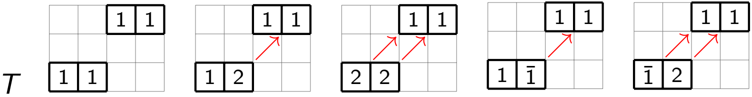

$$\begin{aligned}\mathcal{G}^{\pm}_{\boldsymbol{\nu},\mathbf{b}}(x;t) &= x_1^4 + tx_1^3x_2 + t^2x_1^2x_2^2 - x_1^4 - tx_1^3x_2 \\ &= t^2x_1^2x_2^2\end{aligned}$$

Flagged plethysm

Define *flagged plethysm* $\Pi_{t,x} \colon \mathbb{k}[x_1, \ldots, x_l] \to \mathbb{k}[x_1, \ldots, x_l]$ to be the (over-determined) linear map

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Theorem (Blasiak-Haiman-Morse-Pun-S., 2025+)

$\Pi_{t,x}$ *is well-defined.*

$\Pi_{t,x}$ is a "nonsymmetric analogue" of the plethystic map $f[X] \mapsto f[X/(1-t)]$ for symmetric function f.

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Applications

1. *Modified r-nonsymmetric Macdonald polynomials* (or *modern Macdonald polynomials*) via a flagged HHL formula.
2. A "nonsymmetric shuffle theorem" expressed in terms of flagged LLT polynomials associated to flagged Dyck paths.
3. Tewari-Wilson-Zhang define chromatic nonsymmetric polynomials associated to $d \times d$ Dyck paths starting with r north steps, $\chi_{\mathbf{b},\pi}$. This is (essentially) a signed flagged LLT polynomial.

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Modified Macdonald polynomials

- Plethystically modified Macdonald polynomials
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- $t^{n(\mu)} H_\mu(x; q, t^{-1}) =$ Frobenius series of the Garsia-Haiman module M_μ, a $\mathbb{Q}S_n$-submodule of $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ of dimension $n!$.

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$$H_{31} = ts_4 + (1+qt+q^2t)s_{31} + (q+tq^2)s_{22} + (q+q^2+q^3t)s_{211} + q^3s_{1111}$$

Haglund-Haiman-Loehr formula example

$$H_\mu(X;q,t) = t^{n(\mu)} \sum_R \big(\prod_{\begin{array}{|c|}\hline u \\ \hline \\ \hline \end{array}} q^{a+1} t^{l} \big) \mathcal{G}_R(X;t^{-1})$$

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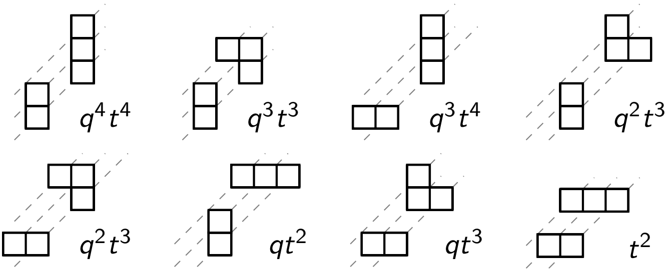
For $\mu = (3,2)$, we get 8 LLT terms:



Haglund-Haiman-Loehr formula example

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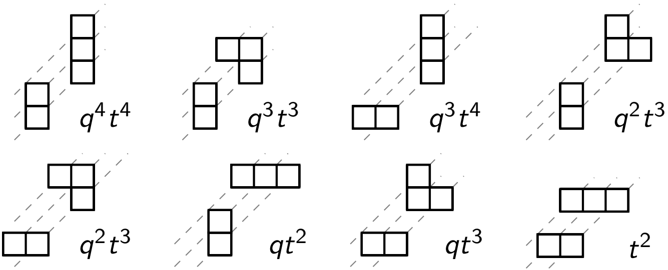

Remark: Can permute the order of ribbons (but changes arm and leg statistics and LLTs.)

Nonsymmetric Macdonald polynomials

The Cherednik operators $Y_1, \ldots, Y_N$ act on $\mathbb{Q}(q,t)[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$.

$$T_i f = s_i f + (1-t)x_i \, \frac{f - s_i f}{x_i - x_{i+1}}\,, \qquad \text{(Demazure-Lusztig operators)}$$

$$\Phi f = f(x_2, \ldots, x_N, qx_1),$$

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- The $\mathcal{E}_\alpha$'s "Hecke symmetrize" to the integral form J_μ's.

Nonsymmetric HHL formula example

$$\mathcal{E}_\alpha(x;q,t) = t^{n(\mu_+)} \sum_R \Big(\prod_{\begin{array}{|c|}\hline u \\ \hline \\ \hline\end{array}} q^{a+1} t^{l} \Big) \mathcal{G}^{\pm}_{R,(1,2,\ldots,m)}(x;t^{-1}).$$

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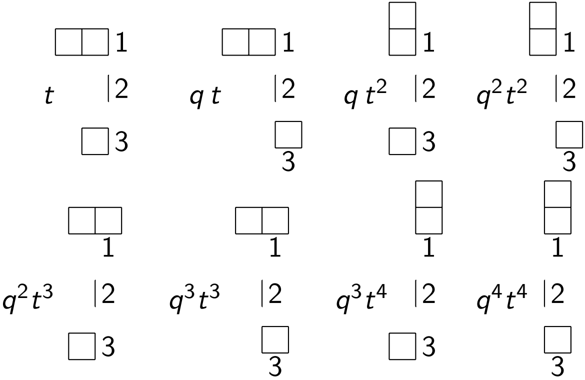
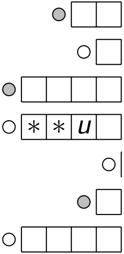
For $\alpha = (2,0,1)$, we get 8 signed flagged LLT terms:



- Convention: bound boxes are always on the same content "diagonal."
- Variation: bound boxes below shape bound shape strictly from below.

Arms and legs



$$a(u) = 2, l(u) = 3$$

The Missing Corner

$$\begin{tikzcd}
\mathcal{E}_\alpha(x_1, \ldots, x_N; q, t) \arrow[r] \arrow[d, "\text{Hecke sym.}"'] & \textbf{?} \arrow[d] \\
J_\mu(x; q, t) \arrow[r, "\text{Plethysm}"'] & H_\mu(x; q, t)
\end{tikzcd}$$

Question

Is there a “nice” family of polynomials that Weyl symmetrizes to the modified Macdonald polynomials?

An important feature for “nice”? Positivity!

Toward positivity

- Knop (2007) formulated a positivity conjecture for a stable version of $\mathcal{E}_\alpha$ involving Kazhdan-Lusztig theory.
- Lapointe (2022) formulated another positivity conjecture for a stable version of $\mathcal{E}_\alpha$.
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- We can understand stabilizations in terms of signed flagged LLT combinatorics.
- Upshot: look at stabilizations for "positivity."

Filling in the missing corner

- $\mathcal{P}(r) = \mathbb{Q}(q,t)[x_1,\ldots,x_r] \otimes \Lambda_{\mathbb{Q}(q,t)}(x_{r+1},\ldots)$.
- Use a "stable" version of $\mathcal{E}_\alpha$, which we denote $\mathcal{J}_{\eta|\lambda}$.

$$\begin{tikzcd}
\mathcal{J}_{\eta|\lambda}(x;q,t) \arrow[r, "\Pi_r"] \arrow[d, "\text{Hecke sym.}"'] & \mathrm{nsH}_{\eta|\lambda}(x;q,t) \arrow[d, "\text{Weyl sym.}"] \\
J_{(\eta;\lambda)_+}(x;q,t) \arrow[r, "\text{Plethysm}"'] & H_{(\eta;\lambda)_+}(x;q,t)
\end{tikzcd}$$

- $\mathcal{J}_{\eta|\lambda}(x;q,t)$ and $\mathrm{nsH}_{\eta|\lambda}(x;q,t)$ for $(\eta|\lambda) \in \mathbb{N}^r \times \mathrm{Par}$ form bases of $\mathcal{P}(r)$.
- $(\eta;\lambda)_+ =$ partition rearrangement of concatenating $(\eta;\lambda)$.
- Π_r is "r-nonsymmetric plethysm"

$$\Pi_r(f(x_1,\ldots,x_r) \otimes g(x)) = (\Pi_{t,x} f(x_1,\ldots,x_r)) \otimes g\left[x/(1-t)\right].$$

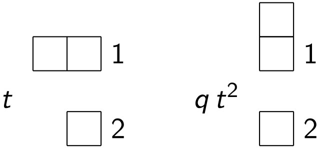
Modern Macdonald polynomials

Definition

The *modern Macdonald polynomial* indexed by $\eta \in \mathbb{N}^r$ and partition λ is

$$\mathrm{nsH}_{\eta|\lambda}(x; q, t) = t^{n((\eta;\lambda)_+)} \sum_{R} \Big(\prod_{\begin{array}{|c|}\hline u \\ \hline \\ \hline \end{array}} q^{a+1} t^{l} \Big) \mathcal{G}_{R,(1,2,\dots,r)}(x; t^{-1})$$

For $r = 2$, $\eta = (2, 1)$, $\lambda = \varnothing$, the LLT terms are



$$\mathrm{nsH}_{21|\varnothing}(x_1, x_2, x_3; q, t) = t\, x_1^3 + x_1^2 x_2 + qt\, x_1^2 x_2 + qt\, x_1 x_2^2 + qt\, x_1^2 x_3 + q\, x_1 x_2 x_3$$

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- $\mathrm{nsH}_{\eta|\lambda}(x; q, t)$ is monomial positive with a combinatorial formula.

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- $\mathrm{nsH}_{\eta|\lambda}(x; q, t)$ form a basis of $\mathcal{P}(r)$.
- We conjecture $\mathrm{nsH}_{\eta|\lambda}(x; q, t)$ is atom positive (would follow from flagged LLTs being atom positive).

Modern Macdonald summary

$$\begin{tikzcd}
\begin{array}{c}\mathcal{E}_{\alpha}(x_1,\ldots,x_N;q,t)=\\ \sum \text{signed flagged LLTs}\end{array} \arrow[d, squiggly, "\text{stabilize}"] & \\
\begin{array}{c}\mathcal{J}_{\eta|\lambda}(x;q,t)=\\ \sum \text{signed flagged LLTs}\end{array} \arrow[r, "\Pi_r"] \arrow[d, "\text{Hecke sym.}"'] & \begin{array}{c}\mathrm{nsH}_{\eta|\lambda}(x;q,t)=\\ \sum \text{flagged LLTs}\end{array} \arrow[d, "\text{Weyl sym.}"] \\
\begin{array}{c}J_{(\eta;\lambda)_+}(x;q,t)=\\ \sum \text{signed LLTs}\end{array} \arrow[r, "\text{Plethysm}"] & \begin{array}{c}H_{(\eta;\lambda)_+}(x;q,t)=\\ \sum \text{LLTs}\end{array}
\end{tikzcd}$$

Extra slides

What follows are extra slides that were not presented.

Stabilization

Definition

A sequence $g_1, g_2, \ldots$ with $g_N \in \mathbb{Q}(q,t)[x_1, \ldots, x_N]$ *converges t-adically* to $f(x) \in \mathcal{P}(r)$ if, for all $e \geq 0$,

$$g_N(x_1, \ldots, x_N) - f(x_1, \ldots, x_N, 0, 0, \ldots)$$

has coefficients whose order of vanishing in t is at least e, for sufficiently large N.

- $1, 1+t, 1+t+t^2, \ldots \to \frac{1}{1-t}$.
- $x_1, tx_1 + x_2, t^2x_1 + x_2 + x_3, t^3x_1 + x_2 + x_3 + x_4, \ldots \to x_2 + x_3 + x_4 + \cdots$.

Stable nonsymmetric Macdonald polynomials

Definition

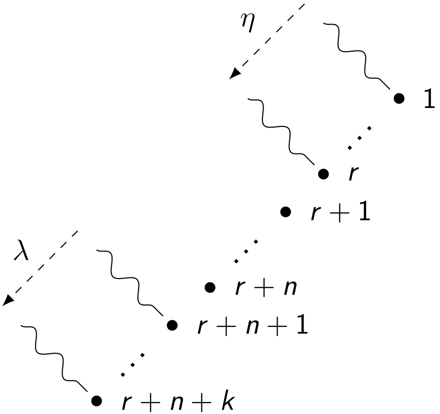
For $(\eta|\lambda) \in \mathbb{N}^r \times \mathrm{Par}$, the *integral form stable r-nonsymmetric Macdonald polynomial* $\mathcal{J}_{\eta|\lambda}(x;q,t) \in \mathcal{P}(r)$ is given by

$$\mathcal{J}_{\eta|\lambda}(x;q,t) = \lim_{n\to\infty} \mathcal{E}_{(\eta;0^n;\lambda)}(x_1,\dots,x_{r+n},0^{\ell(\lambda)};q,t).$$

These $\mathcal{J}_{\eta|\lambda}$'s are integral forms of stable versions of E_α introduced by Bechtloff Weising.

Stable nonsymmetric Macdonald polynomials

Fix some $r \in \mathbb{Z}_{>0}$. Add n empty ribbons after the first r. Set variables $x_{r+n} = x_{r+n+1} = \cdots = 0$.



On (signed) flagged ribbon LLTs, this causes bounds $\geq r+n+1$ to be no restriction and ribbons with bounds below ribbon $\rightarrow 0$.

The plethystic relationship

Theorem (Blasiak-Haiman-Morse-Pun-S., 2025+)

$$\mathcal{J}_{\eta|\lambda}(x;q,t) = t^{n((\eta;\lambda)_+)} \sum_{R} \Big(\prod_{\substack{\boxed{u} \\ \boxed{}}} q^{a+1} t^{l} \Big) \mathcal{G}_{R,(1,2,\ldots,r)}(x;t^{-1})$$

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Theorem (Blasiak-Haiman-Morse-Pun-S., 2025+)

$$\mathrm{nsH}_{\eta|\lambda}(x;q,t) = \Pi_r \mathcal{J}_{\eta|\lambda}(x;q,t)\,.$$

Follows from

- HHL style formula for both sides.
- Stabilized ribbon LLT's satisfy $\mathcal{G}_{R,(1,2,\ldots,r)}(x;t^{-1}) = \Pi_r \mathcal{G}^{\pm}_{R,(1,2,\ldots,r)}(x;t^{-1})$.

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Thank you!