Raising operators in Schubert Calculus

George H. Seelinger (joint with J. Blasiak and J. Morse)

GarsiaFest 2019

ghs9ae@virginia.edu

18 June 2019

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c^{\nu}_{\lambda\mu} = \#$ of points in intersection of subvarieties in a variety X.

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

$$\downarrow$$

Cohomology

Schubert basis $\{\sigma_\lambda\}$ for $H^*(X)$ with property $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^{\nu} \sigma_\nu$

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

$\downarrow$

Cohomology

Schubert basis $\{\sigma_\lambda\}$ for $H^*(X)$ with property $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^{\nu} \sigma_\nu$

$\downarrow$

Representatives

Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^{\nu} f_\nu$

Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

Classical Example

Example

- $X = \mathsf{Gr}_{m,n}$

Classical Example

Example

- $X = \mathrm{Gr}_{m,n}$
- $f_\lambda = s_\lambda$, the Schur functions

Classical Example

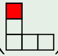
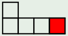
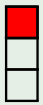
Example

- $X = \mathrm{Gr}_{m,n}$
- $f_\lambda = s_\lambda$, the Schur functions
- $s_\lambda = \prod_{i<j}(1 - R_{ij})h_\lambda$ (Jacobi-Trudi)

Classical Example

Example

- $X = \mathrm{Gr}_{m,n}$
- $f_\lambda = s_\lambda$, the Schur functions
- $s_\lambda = \prod_{i<j}(1 - R_{ij})h_\lambda$ (Jacobi-Trudi)
- Raising operators $R_{i,j}(h_\lambda) = h_{\lambda+\epsilon_i-\epsilon_j}$

$R_{1,3}$ () $=$  $R_{2,3}$ () $=$ 

Schubert Calculus Variations

- There are many variations on classical Schubert calculus of the Grassmannian (Type A)

Schubert Calculus Variations

- There are many variations on classical Schubert calculus of the Grassmannian (Type A)
- (Co)homology of Types BCD Grassmannian, K-theory of Grassmannian, (Co)homology of affine Grassmannian,...

Schubert Calculus Variations

- There are many variations on classical Schubert calculus of the Grassmannian (Type A)
- (Co)homology of Types BCD Grassmannian, K-theory of Grassmannian, (Co)homology of affine Grassmannian,...

Focus

K-theory and K-homology of the affine Grassmannian

Schubert Calculus Variations

- There are many variations on classical Schubert calculus of the Grassmannian (Type A)
- (Co)homology of Types BCD Grassmannian, K-theory of Grassmannian, (Co)homology of affine Grassmannian,. . .

Focus

K-theory and K-homology of the affine Grassmannian

- Simulatenously generalizes K-theory of Grassmannian and (co)homology of affine Grassmannian.

K-Theory of Affine Grassmannian

What is known?

K-Theory of Affine Grassmannian

What is known?

1. K-theory classes of Grassmannian (not affine!) represented by "Grothendieck polynomials." We are interested in their dual:

$$g_\lambda = \prod_{i<j} (1 - R_{ij}) K h_\lambda$$

for Kh_γ an inhomogeneous analogue of h_γ.

K-Theory of Affine Grassmannian

What is known?

1. K-theory classes of Grassmannian (not affine!) represented by "Grothendieck polynomials." We are interested in their dual:

$$g_\lambda = \prod_{i<j}(1 - R_{ij})Kh_\lambda$$

for Kh_γ an inhomogeneous analogue of h_γ.

2. Homology classes of affine Grassmannian represented by k-Schur functions ($t = 1$)

$$s_\lambda^{(k)} = \prod_{(i,j)\in\Delta_\ell^+\setminus\Delta^{(k)}(\lambda)} (1 - R_{ij})h_\lambda$$

K-Theory of Affine Grassmannian

What is known?

1. K-theory classes of Grassmannian (not affine!) represented by "Grothendieck polynomials." We are interested in their dual:

$$g_\lambda = \prod_{i<j}(1 - R_{ij})Kh_\lambda$$

   for Kh_γ an inhomogeneous analogue of h_γ.

2. Homology classes of affine Grassmannian represented by k-Schur functions ($t = 1$)

$$s_\lambda^{(k)} = \prod_{(i,j)\in\Delta_\ell^+\setminus\Delta^{(k)}(\lambda)}(1 - R_{ij})h_\lambda$$

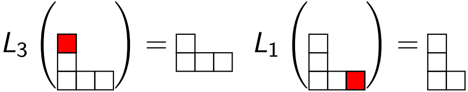
3. (Lam et al., 2010) leave open the question: what is a direct formulation of the K-homology representatives of the affine Grassmannian (K-k-Schur functions)?

Remember?

Goal

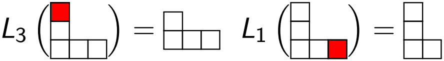
Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

- Lowering Operators $L_j(h_\lambda) = h_{\lambda - \epsilon_j}$



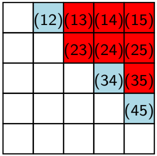
Lowering Operators and Root Ideals

- Lowering Operators $L_j(h_\lambda) = h_{\lambda-\epsilon_j}$



- Root ideal Ψ: given by Dyck path.

$\Psi =$



Roots above Dyck path
Non-roots below

Definition

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) K h_\gamma$$

for Kh_γ an inhomogeneous analogue of h_γ.

Affine K-Theory Representatives with Raising Operators

Definition

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j)\in\mathcal{L}} (1 - L_j) \prod_{(i,j)\in\Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) Kh_\gamma$$

for Kh_γ an inhomogeneous analogue of h_γ.

Example

non-roots of Ψ, roots of $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$K(\Psi; \mathcal{L}; 54332)$$
$$= (1 - L_4)^2 (1 - L_5)^2$$
$$\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) Kh_{54332}$$

Affine K-Theory Representatives with Raising Operators

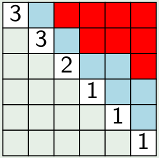
Definition

The *k-Schur root ideal*, $\Delta^{(k)}(\lambda)$ is the unique root ideal with $\lambda_i + \#\text{non-roots in row } i = k$.

Example

$k = 4, \lambda = 332111$

$\Delta^{(4)}(332111) =$



$\leftarrow$ 4 - 2 non-roots

Affine K-Theory Representatives with Raising Operators

Answer (Blasiak-Morse-S., 2019)

Affine K-Theory Representatives with Raising Operators

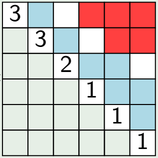
Answer (Blasiak-Morse-S., 2019)

For K-homology of affine Grassmannian,
$f_\lambda = g_\lambda^{(k)} := K(\Delta^{(k)}(\lambda); \Delta^{(k+1)}(\lambda); \lambda)$ since this family satisfies the correct Pieri rule.

Answer (Blasiak-Morse-S., 2019)

For K-homology of affine Grassmannian, $f_\lambda = g_\lambda^{(k)} := K(\Delta^{(k)}(\lambda); \Delta^{(k+1)}(\lambda); \lambda)$ since this family satisfies the correct Pieri rule.

Example

$g_{332111}^{(4)} =$ 

$\Delta_6^+/\Delta^{(4)}(332111), \Delta^{(5)}(332111)$

Property and Further Work

Theorem (Blasiak-Morse-S., 2019)

Property and Further Work

Theorem (Blasiak-Morse-S., 2019)

The $g_\lambda^{(k)}$ are "shift invariant", ie for $\ell = \ell(\lambda)$

$$G_{1^\ell}^{\perp} g_{\lambda+1^\ell}^{(k+1)} = g_\lambda^{(k)}$$

Property and Further Work

Theorem (Blasiak-Morse-S., 2019)

The $g_\lambda^{(k)}$ are "shift invariant", ie for $\ell = \ell(\lambda)$

$$G_{1^\ell}^\perp g_{\lambda+1^\ell}^{(k+1)} = g_\lambda^{(k)}$$

Further work

For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

1. Dual Pieri rule: $G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ??g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ??G_\lambda^{(k)}$, $1 \leq r \leq k$
2. Branching rule: $g_\lambda^{(k)} = \sum_\mu ??g_\mu^{(k+1)}$

References

Thank you!

Blasiak, Jonah, Jennifer Morse, Anna Pun, and Daniel Summers. 2019. *Catalan Functions and k-Schur Positivity*, Journal of the AMS.

Lam, Thomas, Anne Schilling, and Mark Shimozono. 2010. *K-theory Schubert calculus of the affine Grassmannian*, Compositio Math. **146**, 811–852.

Morse, Jennifer. 2011. *Combinatorics of the K-theory of affine Grassmannians*, Advances in Mathematics.