

Raising operators in Schubert Calculus

George H. Seelinger (joint with J. Blasiak and J. Morse)

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ghs9ae@virginia.edu

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Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X .

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Representatives

Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Overview of Schubert Calculus Combinatorics (cont.)

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Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

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- $f_\lambda = s_\lambda$, the Schur functions
- $s_\lambda = \prod_{i < j} (1 - R_{ij}) h_\lambda$ (Jacobi-Trudi)
- Raising operators $R_{i,j}(h_\lambda) = h_{\lambda + \epsilon_i - \epsilon_j}$

$$R_{1,3} \left(\begin{array}{|c|c|c|} \hline \color{red}\square & & \\ \hline \square & & \\ \hline \square & \square & \square \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \color{red}\square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \quad R_{2,3} \left(\begin{array}{|c|} \hline \color{red}\square \\ \hline \square \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline \square & \color{red}\square \\ \hline \square & \square \\ \hline \end{array}$$

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Focus

K -theory and K -homology of the affine Grassmannian

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- (Co)homology of Types BCD Grassmannian, K -theory of Grassmannian, (Co)homology of affine Grassmannian,...

Focus

K -theory and K -homology of the affine Grassmannian

- Simultaneously generalizes K -theory of Grassmannian and (co)homology of affine Grassmannian.

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- 1 K -theory classes of Grassmannian (not affine!) represented by “Grothendieck polynomials.” We are interested in their dual:

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for Kh_γ an inhomogeneous analogue of h_γ .

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- 2 Homology classes of affine Grassmannian represented by k -Schur functions ($t = 1$)

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- 3 (Lam et al., 2010) leave open the question: what is a direct formulation of the K -homology representatives of the affine Grassmannian (K - k -Schur functions)?

Remember?

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Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

Lowering Operators and Root Ideals

- Lowering Operators $L_j(h_\lambda) = h_{\lambda - \epsilon_j}$

$$L_3 \left(\begin{array}{|c|c|c|} \hline \color{red}\square & & \\ \hline \square & & \\ \hline \square & \square & \square \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array} \quad L_1 \left(\begin{array}{|c|c|c|} \hline \square & & \\ \hline \square & & \\ \hline \square & \color{red}\square & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline \square & \\ \hline \square & \\ \hline \square & \\ \hline \end{array}$$

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- Root ideal Ψ : given by Dyck path.

$$\Psi = \begin{array}{|c|c|c|c|c|} \hline & \color{lightblue}{(12)} & \color{red}{(13)} & \color{red}{(14)} & \color{red}{(15)} \\ \hline & & \color{red}{(23)} & \color{red}{(24)} & \color{red}{(25)} \\ \hline & & & \color{lightblue}{(34)} & \color{red}{(35)} \\ \hline & & & & \color{lightblue}{(45)} \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}$$

Roots above Dyck path
Non-roots below

Definition

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) Kh_\gamma$$

for Kh_γ an inhomogeneous analogue of h_γ .

Affine K -Theory Representatives with Raising Operators

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Example

non-roots of Ψ , roots of \mathcal{L}

	(12)	(13)	(14)	(15)
	(23)	(24)	(25)	
		(34)	(35)	
			(45)	

$$\begin{aligned} K(\Psi; \mathcal{L}; 54332) &= (1 - L_4)^2 (1 - L_5)^2 \\ &\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) Kh_{54332} \end{aligned}$$

Affine K -Theory Representatives with Raising Operators

Definition

The k -Schur root ideal, $\Delta^{(k)}(\lambda)$ is the unique root ideal with $\lambda_i + \#\text{non-roots in row } i = k$.

Example

$k = 4, \lambda = 332111$

$$\Delta^{(4)}(332111) = \begin{array}{|c|c|c|c|c|} \hline 3 & & & & \\ \hline & 3 & & & \\ \hline & & 2 & & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline & & & & & 1 \\ \hline \end{array} \leftarrow 4 - 2 \text{ non-roots}$$

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Example

$$g_{332111}^{(4)} = \begin{array}{|c|c|c|c|c|c|} \hline 3 & & & & & \\ \hline & 3 & & & & \\ \hline & & 2 & & & \\ \hline & & & 1 & & \\ \hline & & & & 1 & \\ \hline & & & & & 1 \\ \hline \end{array}$$

$$\Delta_6^+ / \Delta^{(4)}(332111), \Delta^{(5)}(332111)$$

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Further work

For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

- 1 Dual Pieri rule: $G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda G_\lambda^{(k)}$,
 $1 \leq r \leq k$
- 2 Branching rule: $g_\lambda^{(k)} = \sum_\mu g_\mu^{(k+1)}$

Thank you!

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Morse, Jennifer. 2011. *Combinatorics of the K -theory of affine Grassmannians*, Advances in Mathematics.