Schubert calculus and K-theoretic Catalan functions

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UVA Graduate Seminar

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Overview

1. An overview of Schubert calculus
2. Catalan functions: shedding new light on old problems
3. K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

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Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^{\nu} f_\nu$

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Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

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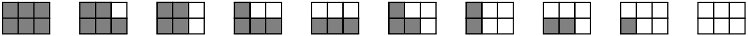

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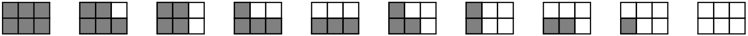

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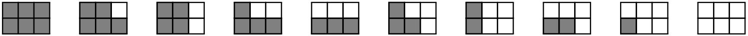

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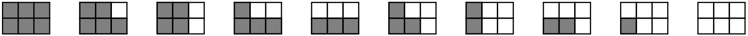

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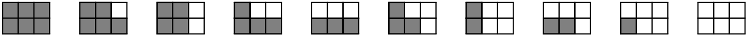

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$c^\nu_{\lambda\mu}$ =number of points in intersection of Schubert varieties.
What are the structure constants $c^\nu_{\lambda\mu}$?

Classical Example (cont.)

$\Lambda_m = \mathbb{C}[x_1, \ldots, x_m]^{S_m}$ is the ring of symmetric polynomials in m variables and has bases indexed by partitions.

$$\underbrace{12x_1^2 + 12x_2^2 - 7x_1x_2}_{\text{symmetric}} \qquad \underbrace{5x_1^2 + 12x_2^2 - 7x_1x_2}_{\text{not symmetric}}$$

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There exists a basis of Λ_m denoted $\{s_\lambda\}_\lambda$ and a surjection of rings such that

$$\begin{aligned} \Lambda_m &\to H^*(\mathrm{Gr}(m,n)) \\ s_\lambda &\mapsto \begin{cases} \sigma_\lambda & \lambda \subseteq (n^m) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Classical Example (cont.)

Cohomology structure: $\sigma_\lambda \leftrightarrow s_\lambda$ when $\lambda \subseteq (n^m)$.

$$s_\lambda s_\mu = \sum_{\nu \subseteq (n^m)} c_{\lambda\mu}^{\nu} s_\nu + \sum_{\nu \not\subseteq (n^m)} c_{\lambda\mu}^{\nu} s_\nu \leftrightarrow \sigma_\lambda \cup \sigma_\mu = \sum_{\nu \subseteq (n^m)} c_{\lambda\mu}^{\nu} \sigma_\nu$$

Schur functions s_λ

Example

Semistandard tableaux: columns increasing and rows non-decreasing.

$$\begin{array}{|c|c|c|c|}\cline{1-1} 5 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 2 & 3 \\ \hline 1 & 2 & 2 & 5 \\ \hline\end{array} \qquad \underbrace{\begin{array}{|c|c|c|c|}\cline{1-1} 8 \\ \cline{1-2} 7 & 9 \\ \cline{1-2} 3 & 4 \\ \hline 1 & 2 & 5 & 6 \\ \hline\end{array}}_{\text{standard } = \text{ no repeated letters}}$$

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Schur function s_λ is a "weight generating function" of semistandard tableaux:

$$\begin{ytableau} 2 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 3 \\ 2 & 2 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 2 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 3 \end{ytableau}, \begin{ytableau} 3 \\ 2 & 3 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 2 \end{ytableau}$$

$$s_{\tiny\yng(1,2)}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + 2x_1x_2x_3$$

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\square} s_{\ydiagram{2,1}} = s_{\ydiagram{3,1}} + s_{\ydiagram{2,2}} + s_{\ydiagram{2,1,1}}$$

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Since $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \ldots, \mu_r)} +$ lower order terms, subtract to get

$$s_{(\mu_1, \ldots, \mu_r)} s_\lambda = \sum c^\nu_{\lambda\mu} s_\nu$$

for well-understood *Littlewood-Richardson coefficients* $c^\nu_{\lambda\mu}$.

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- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$

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- Structure constants $\mathfrak{S}_w \mathfrak{S}_u = c_{wu}^v \mathfrak{S}_v$ are combinatorially unknown.

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(Co)homology of affine Grassmannian	(dual) k-Schur functions
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And many more!

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Upshot

Computations for Schubert polynomials can be moved into symmetric functions.

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- Many definitions. A new one makes proofs easier!

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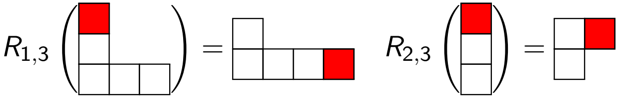

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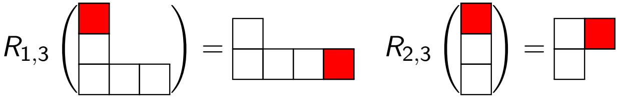

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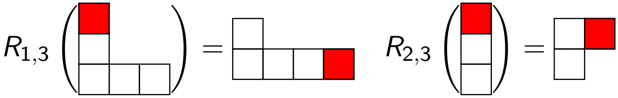

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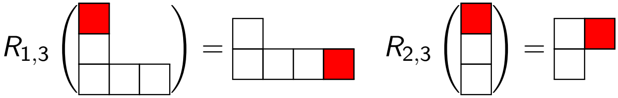

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$$\begin{aligned} s_{22} &= (1 - R_{12}) h_{22} = h_{22} - h_{31} \\ s_{211} &= (1 - R_{12})(1 - R_{23})(1 - R_{13}) h_{211} \\ &= h_{211} - h_{301} - h_{220} - h_{310} + h_{310} + h_{32\,\text{-}1} + h_{400} - h_{41\,\text{-}1} \end{aligned}$$

some terms cancel

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$$s_\alpha = \prod_{i<j}(1 - R_{ij})h_\alpha = \begin{cases} \pm s_\lambda & \text{for a partition } \lambda \\ 0 & \end{cases}$$

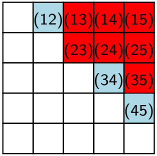
For $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$,

$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1,\ell], |S|=r} s_{\lambda - \epsilon_S}$$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path above the diagonal.

$\Psi =$



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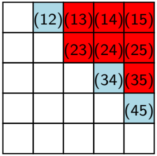
$\Psi =$ 

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Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

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$$H(\Psi;\gamma)(x) = \prod_{(i,j)\in\Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) h_\gamma(x)$$

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- $\Psi = \varnothing \implies H(\varnothing;\gamma) = s_\gamma$
- $\Psi =$ all roots $\implies H(\Psi;\gamma) = h_\gamma$

Catalan functions ($t = 1$)

k-Schur root ideal for λ

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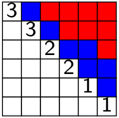
$\Delta^4(3,3,2,2,1,1) =$ 

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Catalan functions ($t = 1$)

k-Schur root ideal for λ

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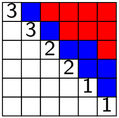
$\Delta^4(3,3,2,2,1,1) =$ 

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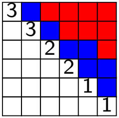
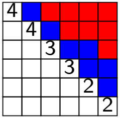
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

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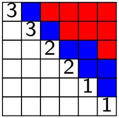
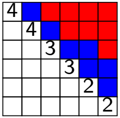
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Branching is a special case of Pieri:

$$s_\lambda^{(k)} = s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda+1^\ell,\mu} s_\mu^{(k+1)}$$

Dual Grothendieck polynomials

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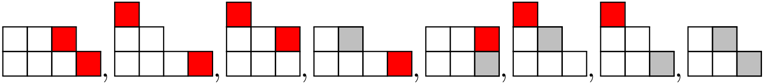

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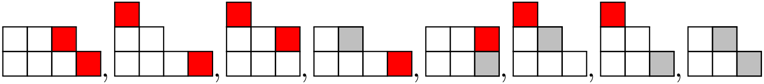

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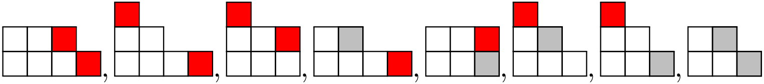

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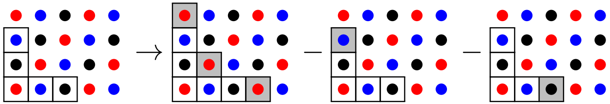

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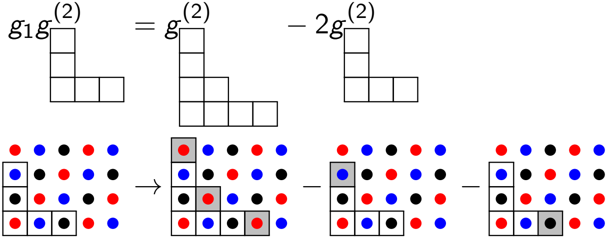

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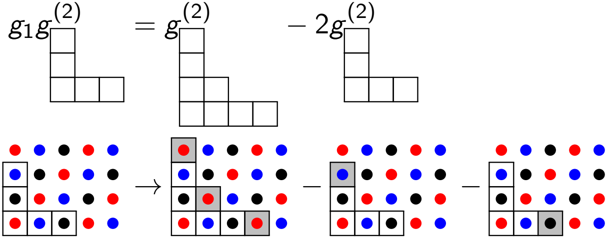

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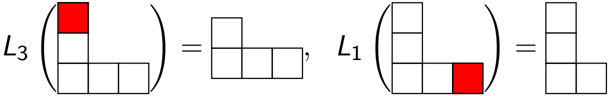
Problem

No direct formula for $g_\lambda^{(k)}$

An Extra Ingredient: Lowering Operators

Lowering Operators $L_j(f_\lambda) = f_{\lambda-\epsilon_j}$



Affine K-Theory Representatives with Raising Operators

K-theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

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Example

non-roots of Ψ, roots of $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$K(\Psi; \mathcal{L}; 54332)$$
$$= (1 - L_4)^2 (1 - L_5)^2$$
$$\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$$

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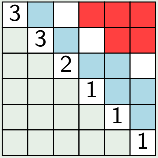
$g^{(4)}_{332111} =$ 

$\Delta_6^+/\Delta^{(4)}(332111)$, $\Delta^{(5)}(332111)$

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satisfy $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

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$$\Phi\colon QK^*(Fl_{k+1}) \to K_*(Gr_{SL_{k+1}})_{loc}$$

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4. Describe the image of $\mathfrak{G}_w^Q$ under Peterson isomorphism for all $w \in S_{k+1}$.

References

Thank you!

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