Flagged LLT Polynomials

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ICERM ECR Seminar

joint work with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun

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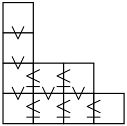
Young Tableaux

Definition

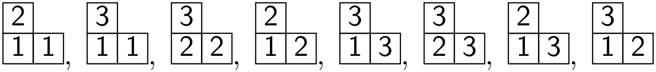
Filling of partition diagram of λ with numbers such that

1. strictly increasing up columns
2. weakly increasing along rows

Collection is called $\mathrm{SSYT}(\lambda)$.



For $\lambda = (2, 1)$,



Schur polynomials

Associate a polynomial to SSYT(λ).

$$\begin{array}{|c|c} \hline 2 & \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c} \hline 3 & \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c} \hline 3 & \\ \hline 2 & 2 \\ \hline \end{array}, \begin{array}{|c|c} \hline 2 & \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c} \hline 3 & \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c} \hline 3 & \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}$$

$$\rightarrow \begin{array}{|c|c} \hline z_2 & \\ \hline z_1 & z_1 \\ \hline \end{array}, \begin{array}{|c|c} \hline z_3 & \\ \hline z_1 & z_1 \\ \hline \end{array}, \begin{array}{|c|c} \hline z_3 & \\ \hline z_2 & z_2 \\ \hline \end{array}, \begin{array}{|c|c} \hline z_2 & \\ \hline z_1 & z_2 \\ \hline \end{array}, \begin{array}{|c|c} \hline z_3 & \\ \hline z_1 & z_3 \\ \hline \end{array}, \begin{array}{|c|c} \hline z_3 & \\ \hline z_2 & z_3 \\ \hline \end{array}, \begin{array}{|c|c} \hline z_2 & \\ \hline z_1 & z_3 \\ \hline \end{array}, \begin{array}{|c|c} \hline z_3 & \\ \hline z_1 & z_2 \\ \hline \end{array}$$

$$s_{(2,1)}(z_1, z_2, z_3) = z_1^2 z_2 + z_1^2 z_3 + z_2^2 z_3 + z_1 z_2^2 + z_1 z_3^2 + z_2 z_3^2 + 2 z_1 z_2 z_3$$

Definition

For λ a partition

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} \boldsymbol{z}^T \text{ for } \boldsymbol{z}^T = \prod_{i \in T} z_i$$

- s_λ is a symmetric function.
- $\{s_\lambda\}_\lambda$ forms a basis for Λ.

Products of Schur polynomials

- Littlewood-Richardson rule: $s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$ for $c^\nu_{\lambda\mu} \in \mathbb{N}$.
- Skew Schur function: $s_{\nu/\lambda} = \sum_{T \in \mathrm{SSYT}(\nu/\lambda)} x^T = \sum_\mu c^\nu_{\lambda\mu} s_\mu$.
- Straightforward observation: $s_{\nu/\lambda} s_{\kappa/\mu} = \sum_{\substack{T \in \mathrm{SSYT}(\nu/\lambda) \\ S \in \mathrm{SSYT}(\kappa/\mu)}} x^T x^S$.
- q-deformation?

Key Object: LLT Polynomials

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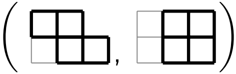
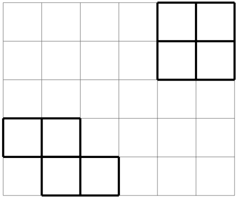
$\boldsymbol{\nu} =$ 



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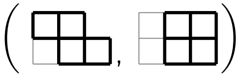
$\boldsymbol{\nu} =$ 

-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

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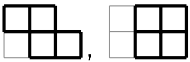
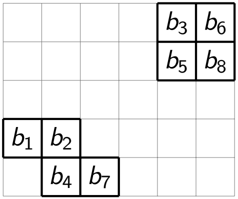
$\boldsymbol{\nu} = \Bigg($  $\Bigg)$



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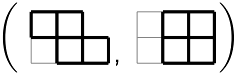
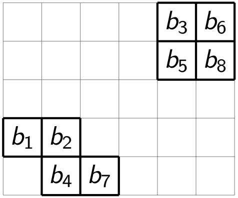
$\boldsymbol{\nu} =$ 



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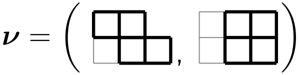

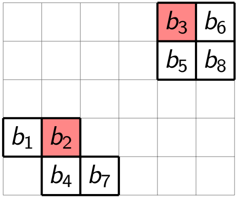

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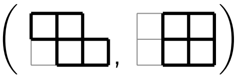
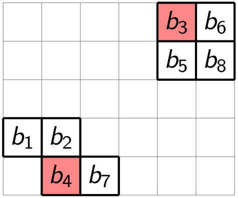
$\boldsymbol{\nu} =$ 



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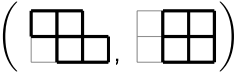
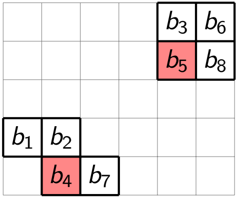
$\boldsymbol{\nu} =$ 



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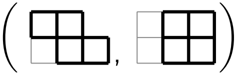
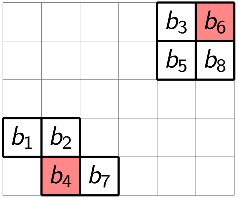
$\boldsymbol{\nu} =$ 



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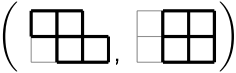
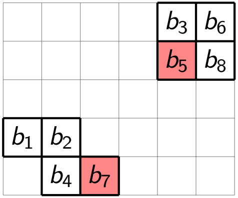
$\boldsymbol{\nu} =$ 



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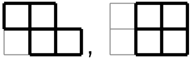
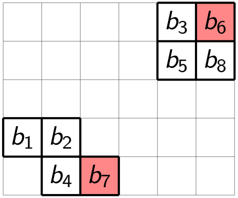
$\boldsymbol{\nu} = \Bigg($  $\Bigg)$



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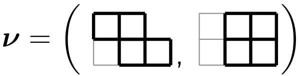

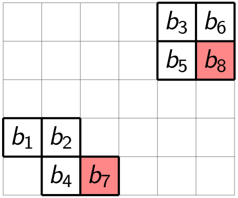

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LLT Polynomials

- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{z}; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} \boldsymbol{z}^T,$$

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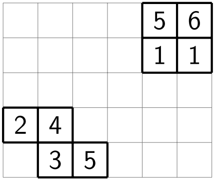

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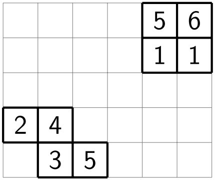

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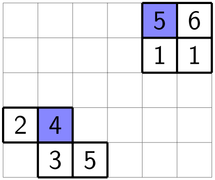

non-inversion

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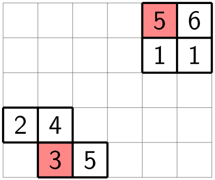

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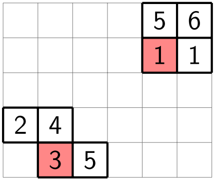

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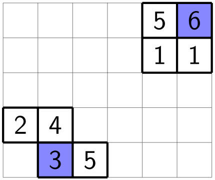

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The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{z}; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{z}^T,$$

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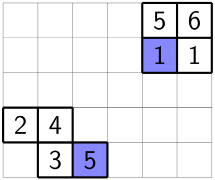

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$$\boldsymbol{z}^T = z_1^2 z_2 z_3 z_4 z_5^2 z_6$$

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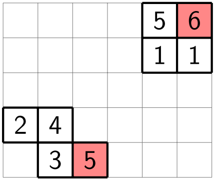

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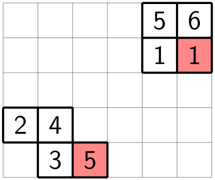

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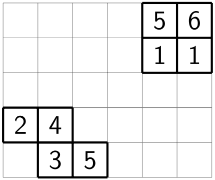

$$\mathrm{inv}(T) = 4, \quad \boldsymbol{z}^T = z_1^2 z_2 z_3 z_4 z_5^2 z_6$$

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- When $\nu^{(i)}$ are partitions, the Schur-expansion coefficients are essentially parabolic Kazdhan-Luzstig polynomials.
- $\mathcal{G}_{\boldsymbol{\nu}}$ is Schur-positive for any tuple of skew shapes $\boldsymbol{\nu}$ [Grojnowski-Haiman, 2007].

Some notable occurrences of LLT Polynomials

- Haglund-Haiman-Loehr formula for Macdonald polynomials:

$$\tilde{H}_{\mu}(X;q,t) = t^{n(\mu)} \sum_{R} \Big(\prod_{\substack{\boxed{u} \\ \boxed{}}} q^{a+1} t^{l} \Big) \mathcal{G}_{R}(X;t^{-1}) \,.$$

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- For $G =$ incomparability graph for natural unit interval order (encoded by Dyck path P),

$$\chi_G(x; t) = (1 - t)^{-|V|} \mathcal{G}_{\nu(P)}[(1 - t)x; t] \,.$$

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- Denote the set of such tableaux via $\mathrm{FT}(\lambda, \mathbf{b})$.
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$$\begin{array}{|c|c}\hline 2 & {\scriptstyle \leq 3} \\ \hline 1 & \multicolumn{1}{|c|}{1} \end{array}{\scriptstyle \leq 1} \qquad \begin{array}{|c|c}\hline 3 & {\scriptstyle \leq 3} \\ \hline 1 & \multicolumn{1}{|c|}{1} \end{array}{\scriptstyle \leq 1}$$

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- Note, there is no issue with letting length of $\mathbf{b}$ be less than l.

Flagged Schur functions (Lascoux-Schützenberger, Wachs)

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- $\mathfrak{S}_w(x) = s_{\lambda,\mathbf{b}}(x)$ for *vexillary* $w \in S_n$ (2143-avoiding).
- $s_{\lambda,\mathbf{b}}(x)$ are examples of *Demazure characters*.

Demazure characters and atoms

The *Demazure operator* π_i acts on $f \in \mathbb{Q}(q,t)[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$ by

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The *Demazure characters* or *key polynomials* are constructed from

- $\mathcal{D}_\lambda = x^\lambda := x_1^{\lambda_1} \cdots x_N^{\lambda_N}$ for partition λ.
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Demazure atoms are defined the same as keys but with $\hat{\pi}_i := \pi_i - 1$ in place of π_i:

- $\mathcal{A}_\lambda = x^\lambda$ for partition λ.
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Examples

$$\mathcal{D}_{520} = x_1^5 x_2^2$$

$$\mathcal{D}_{250} = \pi_1 \mathcal{D}_{520} = \pi_1(x_1^5 x_2^2) = x_1^5 x_2^2 + x_1^4 x_2^3 + x_1^3 x_2^4 + x_1^2 x_2^5$$

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$$\mathcal{D}_{520} = \mathcal{A}_{520}$$
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$$\mathcal{D}_{205} = \mathcal{A}_{520} + \mathcal{A}_{250} + \mathcal{A}_{502} + \mathcal{A}_{205}$$
$$\vdots$$

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- For *Weyl symmetrization operator* π_{w_0}, we have

$$\pi_{w_0}(s_{\lambda,\mathbf{b}}(x)) = s_\lambda(x), \quad \pi_{w_0}(\mathcal{D}_\alpha) = \mathcal{D}_{\mathrm{sort}(\alpha)} = s_{\alpha_+},$$

$$\pi_{w_0}(\mathcal{A}_\alpha) = \begin{cases} s_\alpha & \alpha \text{ a partition,} \\ 0 & \text{else.} \end{cases}$$

Flagged LLT Polynomials (Blasiak-Haiman-Morse-Pun-S.)

- Let $e_1, \ldots, e_l$ be the row ends of $\boldsymbol{\nu}$, ordered in reverse reading order.
- Fix flag $\mathbf{b} = (b_1 \leq \cdots \leq b_l)$.
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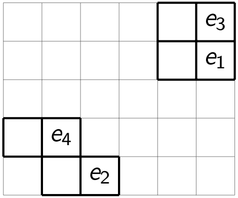

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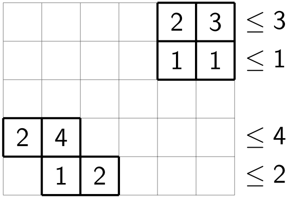

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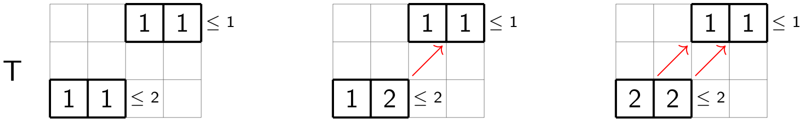

$$\mathcal{G}_{r,\boldsymbol{\nu}}(x;t) = x_1^4 + t\,x_1^3 x_2 + t^2 x_1^2 x_2^2$$

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- $\mathcal{G}_{\boldsymbol{\nu},\mathbf{b}}(x;t)$ are conjecturally positive in terms of *Demazure atoms*.

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The *signed flagged LLT polynomial* indexed by $\boldsymbol{\nu}$ and flag $\mathbf{b}$ is

$$\mathcal{G}^{\pm}_{\boldsymbol{\nu},\mathbf{b}}(\boldsymbol{x}; t) = \sum_{T \in \mathrm{FT}^{\pm}(\boldsymbol{\nu},\mathbf{b})} t^{\mathrm{inv}(T)} (-t)^{-\#\mathrm{bar}(T)} \boldsymbol{x}^{|T|},$$

where $|T|$ is the result of removing all bars from T.

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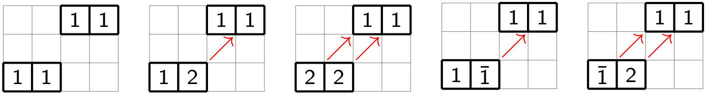
T



$$\begin{aligned} \mathcal{G}^{\pm}_{\boldsymbol{\nu},\mathbf{b}}(\boldsymbol{x};t) &= x_1^4 + t x_1^3 x_2 + t^2 x_1^2 x_2^2 - x_1^4 - t x_1^3 x_2 \\ &= t^2 x_1^2 x_2^2 \end{aligned}$$

Flagged plethysm

Define *flagged plethysm* $\Pi_{t,x} \colon \Bbbk[x_1, \ldots, x_l] \to \Bbbk[x_1, \ldots, x_l]$ to be the (over-determined) linear map

$$\Pi_{t,x} \mathcal{G}^{\pm}_{\boldsymbol{\nu},\mathbf{b}}(x; t^{-1}) = \mathcal{G}_{\boldsymbol{\nu},\mathbf{b}}(x; t^{-1}) \, .$$

Theorem (Blasiak-Haiman-Morse-Pun-S., 2025+)

$\Pi_{t,x}$ *is well-defined.*

Note, $\Pi_{t,x}$ is a "nonsymmetric analogue" of the plethystic map $f[X] \mapsto f[X/(1-t)]$ for symmetric function f.

Applications

1. Can define *modified r-nonsymmetric Macdonald polynomials* (or *modern Macdonald polynomials*) via a *flagged* Haglund-Haiman-Loehr formula:

$$H_{\eta|\lambda}(x;q,t) = t^n \sum_{R} \Big(\prod_{\substack{\boxed{u} \\ \boxed{}}} q^{a+1} t^{l} \Big) \mathcal{G}_{R,(b_1,\ldots,b_r)}(x;t^{-1})\,.$$

   These symmetrize to Macdonald H-polynomials and are conjecturally Demazure atom positive.

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3. Tewari-Wilson-Zhang define chromatic nonsymmetric polynomials associated to $d \times d$ Dyck paths starting with r north steps, $\chi_{\mathbf{b},\pi}$. We show

$$\chi_{(1,\ldots,r),\pi}(x;t) = (1-t)^{r-d} \mathcal{G}^{\pm}_{\nu(\pi),(1,\ldots,r)}(x;t) \,.$$