Diagonal Harmonics and Shuffle Theorems

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joint with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun

ISU Algebra Seminar

27 October 2022

Symmetric Group

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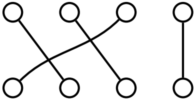
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} =$$ 

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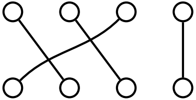

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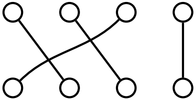

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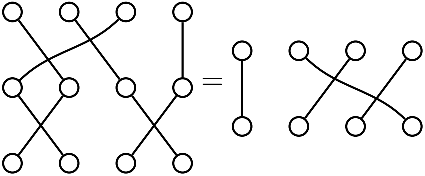

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- $\sigma \in S_n$ acts as $\sigma.f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$

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$$
\begin{aligned}
e_1 = x_1 + x_2 + x_3 = & h_1 \\
e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad & h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\
e_3 = x_1x_2x_3 \quad & h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \cdots
\end{aligned}
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$$\begin{aligned} e_1 &= x_1 + x_2 + x_3 = h_1 \\ e_2 &= x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\ e_3 &= x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \cdots \end{aligned}$$

- $\{f \in \mathbb{Q}[x_1, \ldots, x_n] \mid \sigma.f = f \,\forall \sigma \in S_n\}$ forms a vector space, $\Lambda_{\mathbb{Q}}$.

Combinatorics of Symmetric Polynomials

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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Symmetric functions are polynomials in the $e_1, e_2, \ldots$, or in the $h_1, h_2, \ldots$

$$3h_2 h_1^2 - h_2^2 + 6h_3 h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

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Basis of $\Lambda_{\mathbb{Q}}$?

Partitions

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

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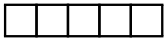
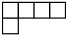
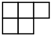
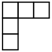
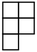
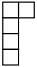
$5 \rightarrow$ 

$4 + 1 \rightarrow$ 

$3 + 2 \rightarrow$ 

$3 + 1 + 1 \rightarrow$ 

$2 + 2 + 1 \rightarrow$ 

$2 + 1 + 1 + 1 \rightarrow$ 

$1 + 1 + 1 + 1 + 1 \rightarrow$ 

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Collection is called $\mathrm{SSYT}(\lambda)$.

For $\lambda = (2, 1)$,

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Schur functions

Associate a polynomial to SSYT(λ).

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Weight:	(2,1,0)	(2,0,1)	(0,2,1)	(1,2,0)	(1,0,2)	(0,1,2)	(1,1,1)	(1,1,1)

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$$s_{(21)}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + 2x_1x_2x_3$$

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$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

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- Schur functions form a basis for $\Lambda_{\mathbb{Q}}$

Why Schur functions?

Harmonic polynomials

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Explicitly, for

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$$\begin{aligned} M &= \mathrm{sp}\left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\qquad x_3 - x_1, x_2 - x_3, 1\} \end{aligned}$$

Harmonic polynomials

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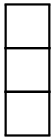


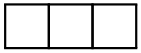

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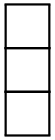


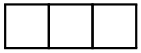

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$$\underbrace{\mathrm{sp}\{\Delta\}}_{(1,1,1)} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{(2,1)} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{(2,1)} \oplus \underbrace{\mathrm{sp}\{1\}}_{(3)}$$

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$$e_1^3 = (x_1+x_2+x_3)^3 = s_{(1,1,1)} + s_{(2,1)} + s_{(2,1)} + s_{(3)}$$

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3. How many times does an S_n fixed subspace occur? Frobenius:

$$e_1^3 = (x_1+x_2+x_3)^3 = s_{\ydiagram{1,1,1}} + s_{\ydiagram{2,1}} + s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

Schur basis expansion counts multiplicity of irreducible S_n fixed subspaces!

Recap so far

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Upshot

Via Frobenius characteristic map, questions about S_n-action on vector spaces (representations) get translated to questions about Schur expansion coefficients in symmetric functions.

Does a symmetric function expand into Schur basis with nonnegative coefficients? Is there a combinatorial description for coefficients?

Getting more information

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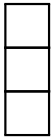
Break M up into smallest S_n fixed subspaces

$$\underbrace{\mathrm{sp}\{\Delta\}}\oplus\underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\,2x_2(x_3-x_1)-x_3^2+x_1^2\}}\oplus\underbrace{\mathrm{sp}\{x_3-x_1,\,x_2-x_3\}}\oplus\underbrace{\mathrm{sp}\{1\}}$$



deg=2

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Solution: minimal S_n-fixed subspace of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\ydiagram{1,1,1}} + q^2 s_{\ydiagram{2,1}} + q s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

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Answer: "Hall-Littlewood polynomial" $H_{(1,1,1)}(X;q)$.

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- $\tilde{H}_\lambda(X;1,1) = e_1^{|\lambda|}$.
- Does there exist a family of S_n-representations whose (bigraded) Frobenius characteristics equal $\tilde{H}_\lambda(X;q,t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

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- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$. (Still open!)

Garsia-Haiman modules

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \mathrm{sp}\{f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \mid \left(\sum_{j=1}^{n} \partial_{x_j}^r \partial_{y_j}^s\right) f = 0, \forall r + s > 0\}$$

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Question

What symmetric function is the bigraded Frobenius characteristic of DH_n?

∇e_n

Frobenius characteristic of DH_3

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$$= \frac{t^3\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2t+qt^2+qt)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{q^3\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

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Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q+t+1)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

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Operator ∇

$$\nabla \tilde{H}_\lambda(X; q, t) = q^{n(\lambda)} t^{n(\lambda')} \tilde{H}_\lambda(X; q, t)$$

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Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n.

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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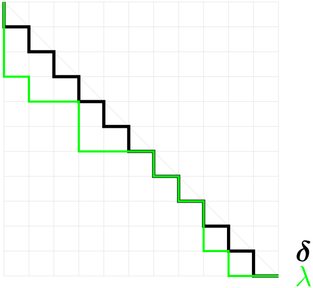
Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



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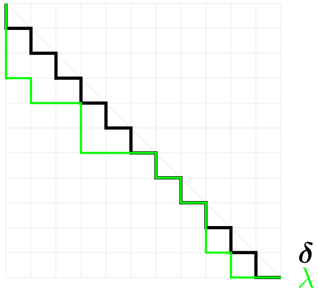

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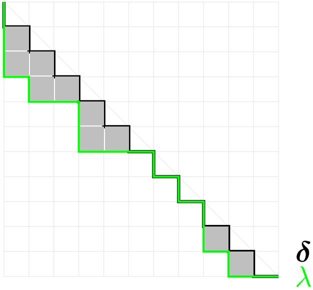

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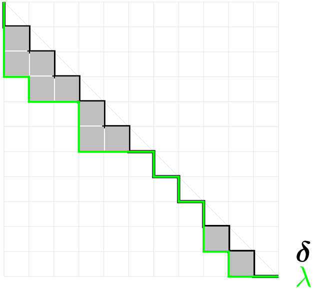

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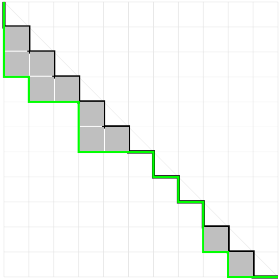

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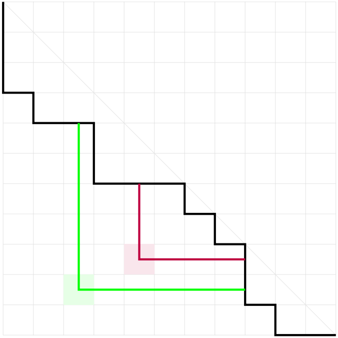
dinv

$\text{dinv}(\lambda) = \#$ of balanced hooks in diagram below λ.



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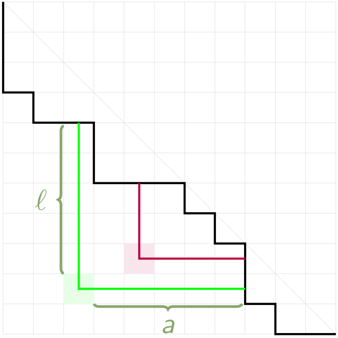

Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

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- $\mathcal{G}_\nu$ is Schur-positive for any tuple of skew shapes ν [Grojnowski-Haiman, 2007].

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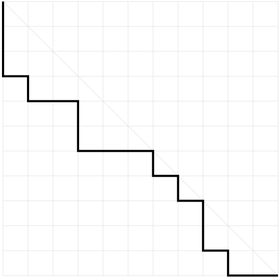

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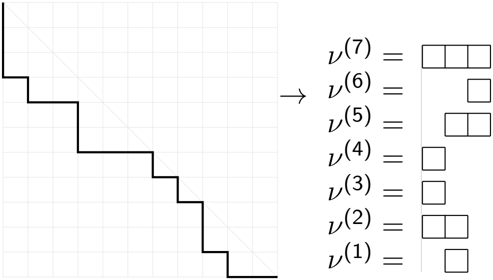

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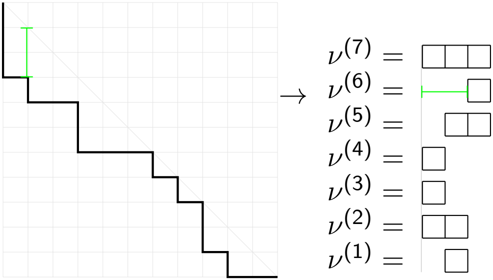

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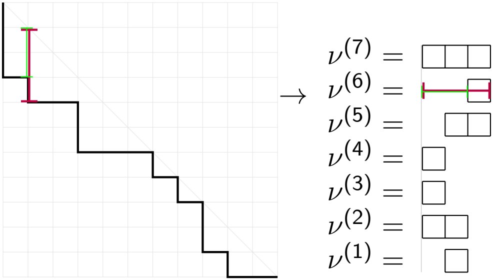

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- $$\mathcal{G}_{\square\square,\square}(x_1,x_2;q) = x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3 + qx_1^2x_2 + qx_1x_2^2$$

1 1	1 2	1 2	2 2	1 1	2 2
1	1	2	2	2	1

$$= s_3 + qs_{2,1}$$

Example ∇e_3

λ	$q^{\mathsf{dinv}(\lambda)} t^{\mathsf{area}(\lambda)}$	$q^{\mathsf{dinv}(\lambda)} t^{\mathsf{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$

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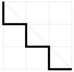		
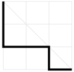		
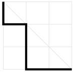		
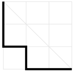		
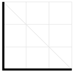		

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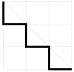
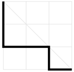
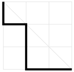
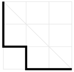
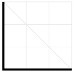
	q^3	
	$q^2 t$	
	qt	
	qt^2	
	t^3	

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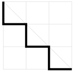
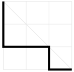
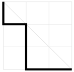
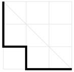
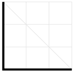
	q^3	$s_3 + q s_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$q t s_{2,1} + q^2 t s_{1,1,1}$
	qt	$t s_{2,1} + q t s_{1,1,1}$
	qt^2	$t^2 s_{2,1} + q t^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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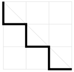
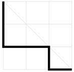
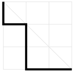
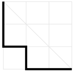
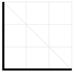
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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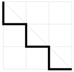
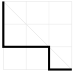
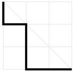
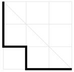
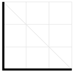
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

- Entire quantity is q, t-symmetric
- Coefficient of $s_{1,1,1}$ in sum is a "(q, t)-Catalan number" $(q^3 + q^2 t + qt + qt^2 + t^3)$.

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For m, n coprime, the operator $e_k[-MX^{m,n}]$ acting on Λ satisfies

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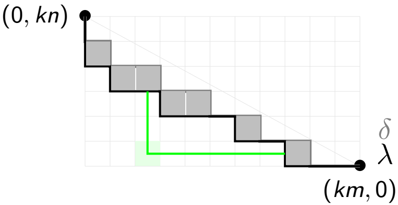

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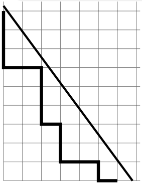

area(λ) as before
$\text{dinv}_p(\lambda) = \#p$-balanced hooks $\frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

Proof Overview (algebraic side)

- $\psi\colon \mathcal{E}^+ \cong S$
- $\mathcal{E}^+$ is the "positive half" of $\mathcal{E}$
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Key relationship

For $\xi \in \mathcal{E}^+$,

$$\omega(\xi \cdot 1) = \mathrm{pol}_X(\psi(\xi))$$

for automorphism $\omega\colon \Lambda \to \Lambda$ and $\mathrm{pol}_X\colon S \to \Lambda$ a "polynomial truncation" operation.

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- For $\xi = D_{\mathbf{b}}$, we get

$$\operatorname{pol}_X \mathbf{H}_q \left(\frac{z^{\mathbf{b}} \prod_{i<j+1} (1 - qtz_i/z_j)}{\prod_{i<j} (1 - tz_i/z_j)} \right) = \sum_{\lambda} t^{\mathsf{area}(\lambda)} q^{\mathsf{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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$$z^{\mathbf{b}} \frac{\prod_{i<j+1}(1 - qtz_i/z_j)}{\prod_{i<j}(1 - tz_i/z_j)} \qquad = \qquad ??$$

$$\downarrow \qquad\qquad \downarrow$$

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$$\downarrow \qquad\qquad\qquad \downarrow$$

$$D_{\mathbf{b}} \cdot 1 \quad = \sum\nolimits_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

Need an "infinite series" version of LLT polynomials!

Cauchy identity

- For a fixed $\sigma \in S_l$, there exists a basis of $\mathbb{Q}(q)[z_1^{\pm 1}, \ldots, z_l^{\pm 1}]$ called "non-symmetric Hall-Littlewood polynomials", denoted $E_\lambda^\sigma = E_\lambda^\sigma(z_1, \ldots, z_l; q)$ for $\lambda \in \mathbb{Z}^l$.

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- Under an inner-product coming from representation theory of affine Hecke algebras, there is a dual basis $F_\lambda^\sigma = E_{-\lambda}^{\sigma w_0}(z_1^{-1}, \ldots, z_l^{-1}; q^{-1}) = \overline{E_{-\lambda}^{\sigma w_0}}$

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- (Grojnowski-Haiman 2007) defines a (symmetric) "series LLT" polynomial $\mathcal{L}_{\beta/\alpha}^\sigma(x_1, \ldots, x_l; q) = H_q(w_0(F_\beta^{\sigma^{-1}} \overline{E_\alpha^{\sigma^{-1}}}))$

Proof Idea

Stable Shuffle Theorem (BHMPS 21a)

For $\mathbf{b} \in \mathbb{Z}^l$ corresponding to highest path under a line of slope $-r/s$,

$$\psi D_{\mathbf{b}} = \sum_{a_1,\ldots,a_{l-1}\geq 0} t^{|\mathbf{a}|} \mathcal{L}^{\sigma}_{((b_l,\ldots,b_1)+(0,a_{l-1},\ldots,a_1))/(a_{l-1},\ldots,a_1,0)}(x_1,\ldots,x_l;q)$$

Under polynomial truncation,
$\mathcal{L}^{\sigma}_{\beta/\alpha}(x_1,\ldots,x_l;q) \to q^{\mathrm{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)}(x_1,\ldots,x_l;q^{-1})$

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$$\implies \omega(D_{\mathbf{b}}\cdot 1)(x_1,\ldots,x_l) = \sum_{\lambda} t^{\mathrm{area}(\lambda)}q^{\mathrm{dinv}_p(\lambda)}\mathcal{G}_{\nu(\lambda)}(x_1,\ldots,x_l;q^{-1}).$$

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$$\Delta_{h_r}\Delta'_{e_{n-1}} e_k = \langle z^n \rangle \sum_{\lambda,P} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(\lambda)} x^P \prod_{r_i(\lambda)=r_{i-1}(\lambda)+1} \left(1 + zt^{-r_i(\lambda)}\right).$$

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Loehr-Warrington Conjecture (2008)

$$\nabla s_\mu = \operatorname{sgn}(\mu) \sum_{(G,R) \in LNDP_\mu} t^{\operatorname{area}(G,R)} q^{\operatorname{dinv}(G,R)} x^R$$

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Generalizing our methods further, we arrive at the following.

Theorem (BHMPS21c)

$$s_\mu[-MX^{m,n}] \cdot 1 = \sum_\pi t^{a(\pi)} q^{\mathrm{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1})$$

For a sum over all “nests” π in a “den” corresponding to s_μ.

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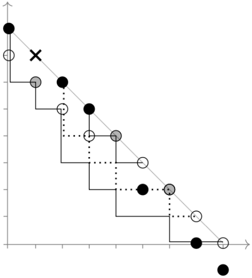

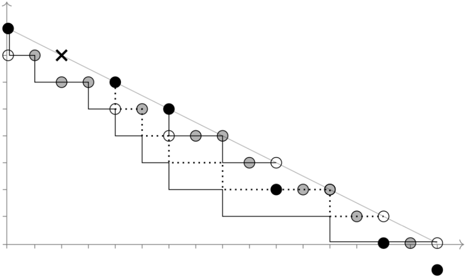

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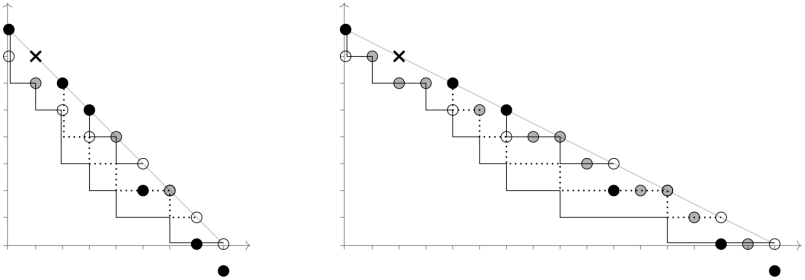

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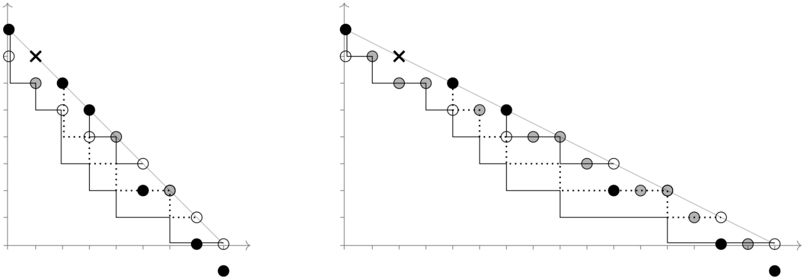

- Implies the Loehr-Warrington Conjecture as a special case.
- Also proves $\mathrm{sgn}(\mu)\nabla s_\mu$ is Schur-positive.

Generalizations

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Convex Curve Conjecture (Blasiak-Haiman-Morse-Pun-S., 2021a)

For $\mathbf{b} = (b_1, \ldots, b_l)$ the south steps of highest path under a convex curve, the Schur expansion of $D_{\mathbf{b}} \cdot 1$ has coefficients in $\mathbb{N}[q, t]$.

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- Experimental computation suggests this is "tight."
- Coefficient of $s_{1,\ldots,1}$ coincides with (q, t)-polynomials found in (Gorsky-Hawkes-Schilling-Rainbolt, 2020), (Galashin-Lam, 2021).

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- What are the Schur expansion coefficients of $D_{\mathbf{b}} \cdot 1$?

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- What can we say about Macdonald polynomials?
- S_l-representation theory interpretations?

References

Thank you!

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