k-SCHUR FUNCTIONS AS SCHUBERT REPRESENTATIVES FOR THE AFFINE GRASSMANNIAN

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1. Introduction

Classically, the Grassmannian of m-planes in $\mathbb{C}^{m+n}$, denoted $X = \mathrm{Gr}(m,n)$, has a decomposition into Schubert cells, Ω_λ for partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda_i \leq n$. One can then define Schubert varieties $X_\lambda = \overline{\Omega_\lambda}$, the closure of a Schubert cell. In turn, this gives a basis for the cohomology ring $H^*(X) = \bigoplus_\lambda \mathbb{Z}\sigma_\lambda$ for σ_λ a representative of X_λ. Then, the coefficients in the product of $\sigma_\lambda \sigma_\mu$ expressed as a linear combination of Schubert representatives contain useful geometric information.

On the other hand, the ring of symmetric functions $\Lambda = \mathbb{Z}[h_1(x), h_2(x), \ldots]$ for $h_d(x) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_d} x_{i_1} x_{i_2} \cdots x_{i_d}$ has a distinguished basis of Schur functions indexed by partitions λ given by

$$s_\lambda = \det(h_{\lambda_i + j - i})_{1 \leq i,j \leq \ell}$$

where $h_0(x) = 1$ and $h_r = 0$ for $r < 0$ by convention. The "Littlewood-Richardson coefficients" $c_{\lambda\mu}^{\nu}$ in the expansion $s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$ have many combinatorial formulas and are well-studied. Thus, the following theorem is quite useful for connecting geometry and combinatorics.

1.1. **Theorem.** *There is a surjection of rings $f \colon \Lambda \to H^*(\mathrm{Gr}(m,n))$ given by*

$$f(s_\lambda) = \begin{cases} \sigma_\lambda & \text{if } \lambda \subseteq (\underbrace{n, \ldots, n}_{m \text{ times}}) \\ 0 & \text{else} \end{cases}$$

In this presentation, we seek to summarize how a special class of symmetric functions called "k-Schur functions", denoted $s_\lambda^{(k)}(x)$, play a similar role for the affine Grassmannian of SL_{k+1} and then explore their combinatorics. We will make precise and summarize the original proof of [Lam08] that k-Schur functions are the Schubert representatives of the affine Grassmannian in Sections 2–4, culminating in Theorem 4.1. Then, in Section 5, we will present a more combinatorial treatment of k-Schur functions. These two parts can be read more-or-less independently.

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2. Preliminaries

2.1. **Definition.** The following is a summary of definitions in [Lam08, Section 2.1].

- Let W be a crystallographic Coxeter group with simple generators $\{s_i \mid i \in I\}$.
- Let Φ be the root system for W
- Let Φ^+ be the positive roots and $\{\alpha_i \mid i \in I\}$ be the simple roots.
- Let $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ be the root lattice and $Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee$ be the co-root lattice.
- Let $\mathfrak{h}_\mathbb{Z}^*$ and $\mathfrak{h}_\mathbb{Z}$ be the weight and co-weight lattice.
- Let $\langle \cdot, \cdot \rangle \colon \mathfrak{h}_\mathbb{Z} \times \mathfrak{h}_\mathbb{Z}^* \to \mathbb{Z}$ be the pairing between $\mathfrak{h}_\mathbb{Z}$ and $\mathfrak{h}_\mathbb{Z}^*$
- Let $W_{\text{aff}} = W \ltimes Q^\vee$ be the affine Weyl group with additional generator s_0.
- For $\lambda \in Q^\vee$, let $t_\lambda \in W_{\text{aff}}$ be the corresponding translation element. Note $t_\lambda \cdot t_\mu = t_{\lambda+\mu}$ and $wt_\lambda w^{-1} = t_{w\cdot\lambda}$ for $w \in W$.
- Let $\ell \colon W_{\text{aff}} \to \mathbb{N}_{\geq 0}$ be the length function for W_{aff}.
- Let W^0 be the minimal length coset representatives of W_{aff}/W, called *Grassmannian* elements.

2.2. **Proposition.** *There exists a natural bijection between W^0 and $Q^\vee$.*

Proof. Each coset of W_{aff}/W contains a unique element of $Q^\vee$ and a unique minimal length coset representative. □

2.1. The affine Grassmannian.

Let G be a simple and simply connected complex algebraic group with Weyl group W. Let K be a maximal compact subgroup and let T be a maximal torus in K. Let $\mathfrak{h}_\mathbb{Z}^\vee$ be the weight lattice and $\mathfrak{h}_\mathbb{Z}$ be the co-weight lattice of T. In our case, we are primarily interested in type A with $G = SL_n$, $W = S_n$, $K = SU_n$, $T = (\mathbb{C}^\times)^n$, $W_{\text{aff}} = \tilde{S}_n$ is the affine symmetric group, and affine simple roots are given by $\{\alpha_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$.

For $F = \mathbb{C}((t))$ and $O = \mathbb{C}[\![t]\!]$, we define $\mathrm{Gr} = \mathrm{Gr}_G = G(F)/G(O)$ to be the affine Grassmannian.

2.3. **Proposition.** Gr *is homotopy equivalent to ΩK. Thus, the two spaces have isomorphic (co)homology theories.*

Let $S = \mathrm{Sym}(\mathfrak{h}_\mathbb{Z}^*)$ be the symmetric algebra of $\mathfrak{h}_\mathbb{Z}^*$. From my last presentation, we know that $G(F)$ has Bruhat decomposition

$$G(F) = \bigcup_{w \in W_{\text{aff}}} \mathcal{B}w\mathcal{B}$$

for Iwahori subgroup $\mathcal{B}$. This induces a decomposition of Gr into Schubert cells $\Omega_w = \mathcal{B}wG(O) \subseteq G(K)/G(O)$:

$$\mathrm{Gr} = \bigsqcup_{w \in W^0} \Omega_w = \bigcup_{w \in W^0} X_w$$

where $X_w = \overline{\Omega_w}$ are Schubert varieties. Then, we define the following

2.4. **Definition.** We denote the following Schubert classes in homology, cohomology, equivariant homology, and equivariant cohomology as follows.

(a) Let $\sigma_w \in H_*(\mathrm{Gr})$;
(b) Let $\sigma^w \in H^*(\mathrm{Gr})$;
(c) Let $\sigma_{(w)} \in H_T(\mathrm{Gr})$, the T-equivariant homology of Gr;
(d) Let $\sigma^{(w)} \in H^T(\mathrm{Gr})$, the T-equivariant cohomology of Gr.

2.5. **Remark.** Note that $S = \mathrm{Sym}(\mathfrak{h}_{\mathbb{Z}}^*) = H^T(\mathrm{pt})$.

3. Four Hopf Algebras

To show that (dual) k-Schur functions are Schubert representatives for the (co)homology of the affine Grassmannian, we will string together various results concerning four different Hopf algebras.

3.1. The affine nilHecke ring.

3.1. **Definition.** (a) Define $\mathbb{A}_{\mathrm{aff}}$ to be the ring with 1 over $\mathbb{Z}$ given by generators $\{A_i \mid i \in I \cup \{0\}\} \cup \{\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*\}$ and relations

$$\begin{aligned} A_i\lambda &= (s_i \cdot \lambda)A_i + \langle \lambda, \alpha_i^\vee \rangle \cdot 1 && \text{for } \lambda \in \mathfrak{h}_{\mathbb{Z}}^* \\ A_iA_i &= 0 \\ (A_iA_j)^m &= (A_jA_i)^m && \text{if } (s_is_j)^m = (s_js_i)^m \end{aligned}$$

where the elements of $\mathfrak{h}_{\mathbb{Z}}^*$ commute with each other.

(b) For $w \in W_{\mathrm{aff}}$, let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for w. Then, we define $A_w := A_{i_1} \cdots A_{i_\ell}$.
(c) Let $\mathbb{A}_0 = \mathbb{Z}[A_i \mid i \in I \cup \{0\}] \subseteq \mathbb{A}_{\mathrm{aff}}$ be the *affine nilCoxeter algebra* generated by the A_i's.

3.2. **Proposition** ([KK86]). *$\{A_w \mid w \in W_{aff}\}$ is an S-basis of $\mathbb{A}_{aff}$.*

There is a specialization map $\phi_0 \colon \mathbb{A}_{\mathrm{aff}} \to \mathbb{A}_0$ given by

$$\phi_0\left(\sum_w a_wA_w\right) = \sum_w \phi_0(a_w)A_w$$

where $\phi_0(s)$ evaluates the polynomial $s \in S = \mathrm{Sym}(\mathfrak{h}_{\mathbb{Z}}^*)$ at 0. (Recall elements in $\mathfrak{h}_{\mathbb{Z}}^*$ are maps from $\mathfrak{h}_{\mathbb{Z}}$ to $\mathbb{Z}$.)

3.3. **Proposition.** *$\mathbb{A}_{aff}$ is a Hopf algebra with coprodcut map $\Delta \colon \mathbb{A}_{aff} \to \mathbb{A}_{aff} \otimes_S \mathbb{A}_{aff}$ given by*

$$\begin{aligned} \Delta(s) &= 1 \otimes s = s \otimes 1 && s \in S \\ \Delta(A_i) &= A_i \otimes 1 + 1 \otimes A_i - A_i \otimes \alpha_iA_i \end{aligned}$$

3.2. Hopf algebra structure of $H_T(\Omega K)$ and Peterson's j-Homomorphism.

Over Frac(S), $H_T(\Omega K)$ is spanned by the classes $\{\psi_t \mid t = t_\lambda \in Q^\vee \subseteq W_{\text{aff}}\}$. Since ΩK is a group with T-equivariant multiplications,

3.4. **Proposition.** *$H_T(\Omega K)$ and $H^T(\Omega K)$ have the structure of dal Hopf algebras. Furthermore, $H_T(\Omega K)$ has Hopf algebra structure given by*

$$\psi_{id} = 1 \qquad \epsilon(\psi_t) = 1 \qquad c(\psi_t) = \psi_{t^{-1}}$$
$$\omega(\psi_t) = \psi_t \otimes \psi_t \qquad \psi_t \psi_{t'} = \psi_{tt'}$$

with scalars $s \in S \subseteq H_T(\Omega K)$ central and $c(s) = s$. In particular, $H_T(\Omega K)$ is commutative and co-commutative as a Hopf algebra.

3.5. **Definition.** Let the *Peterson subalgebra* be $Z_{\mathbb{A}_{\text{aff}}}(S)$, the centralizer of S in $\mathbb{A}_{\text{aff}}$.

3.6. **Proposition.** *There is a Hopf algebra isomorphism $j\colon H_T(\Omega K) \to Z_{\mathbb{A}_{aff}}(S)$ sending classes $\psi_{t_\lambda} \mapsto t_\lambda$ where $t_\lambda \in Q^\vee \subseteq W_{aff}$.*

3.3. Affine Fomin-Stanley subalgebra.

3.7. **Definition.** The *affine Fomin Stanley subalgebra* is the subalgebra

$$\mathbb{B}' = \{a \in \mathbb{A}_0 \mid \phi_0(as) = \phi_0(s)a \text{ for all } s \in S\}$$

The algebra $\mathbb{B}'$ is a model for the homology $H_*(\mathrm{Gr})$ in the following sense.

3.8. **Proposition.** *Granting that $\phi_0(j(\sigma_{(u)})) \in \mathbb{B}'$,*

(a) The map

$$H_*(\mathrm{Gr}) \to \mathbb{B}'$$
$$\sigma_u \mapsto \phi_0(j(\sigma_{(u)}))$$

is an isomorphism of Hopf algebras.

(b) $\phi_0(j(\sigma_{(u)}))$ is the unique element in $\mathbb{B}'$ with unique Grassmannian term A_u.

3.9. **Corollary.** *$\mathbb{B}'$ is a commutative algebra.*

3.4. Combiatorial affine Fomin-Stanley subalgebra and symmetric functions.

From now on, we will restrict ourselves to type A.

3.10. **Definition.** We construct the combinatorial affine Fomin-Stanley subalgebra:

(a) We say a word a in alphabet $\mathbb{Z}/n\mathbb{Z}$ is *cyclically decreasing* if no letter is repeated and, if $i, i+1$ both occur in a, then $i+1$ occurs to the west of i.

(b) Define $h_i \in \mathbb{A}_0 \subseteq \mathbb{A}_{\text{aff}}$ for $i \in \{0, 1, \ldots, n-1\}$ by

$$h_i = \sum_{\substack{w \in W_{\text{aff}} \\ w \text{ cyclically decreasing} \\ \ell(w) = i}} A_w$$

(c) Let $\mathbb{B} = \langle h_i \mid i \in [0, n-1] \rangle$ be the *combinatorial affine Fomin-Stanley subalgebra* of $\mathbb{A}_0$.

3.11. **Example.** The word $s_1 s_0 s_2 \in \tilde{S}_3$ is cyclically decreasing but $s_1 s_2 s_0 \in \tilde{S}_3$ is not. We also have that, for $n = 4$,

$$h_2 = A_3 A_2 + A_3 A_1 + A_0 A_3 + A_2 A_1 + A_2 A_0 + A_1 A_0$$

3.12. **Proposition.** *$\mathbb{B}$ is commutative and isomorphic to $\Lambda_n = \mathbb{Z}[h_1, \ldots, h_{n-1}]$ via*

$$\begin{aligned} \psi \colon \Lambda_n &\to \mathbb{B} \\ h_i(x) &\mapsto h_i \end{aligned}$$

3.13. **Definition.** Let $\langle \cdot, \cdot \rangle \colon \mathbb{A}_0 \times \mathbb{A}_0 \to \mathbb{Z}$ be given by $\langle A_w, A_v \rangle = \delta_{w,v}$. We define the following symmetric functions.

(a) Let $w \in W_{\text{aff}}$. We define the *affine Stanley symmetric functions* $\tilde{F}_w(x) \in \Lambda$ by

$$\tilde{F}_w(x) = \sum_{a = (a_1, a_2, \ldots, a_t)} \langle h_{a_t} h_{a_{t-1}} \cdots h_{a_1} \cdot 1, A_w \rangle x_1^{a_1} x_2^{a_2} \cdots x_t^{a_t}$$

where the sum is over compositions of $\ell(w)$ satisfying $a_i \in [0, n-1]$.

(b) Define the image of $\{\tilde{F}_w(x) \mid w \in W^0\}$ in the quotient $\Lambda^n = \Lambda / \langle m_\lambda(x) \mid \lambda_1 \geq n \rangle$ to be the *affine Schur functions* (where $m_\lambda(x)$ is the monomial symmetric function indexed by λ).

(c) The *k-Schur functions* $\{s_w^{(k)}(x) \mid w \in W^0\}$ are the dual basis of Λ_n to the affine Schur functions under the Hall inner product where $k = n - 1$.

(d) Let the *non-commutative k-Schur functions* be given by

$$s_w^{(k)} := \psi(s_w^{(k)}(x)) \in \mathbb{B}$$

3.14. **Remark.** Affine Stanley symmetric functions "enumerate" all the ways of factoring reduced words for $w \in W_{\text{aff}}$ into groups of "cyclically decreasing" words. In particular, the coefficient of $x_1 \cdots x_\ell$ gives the number of reduced words for $w \in W_{\text{aff}}$.

3.15. **Example.** For $n = 3 = k + 1$, we compute some non-commutative k-Schur functions.

$$\begin{aligned} s_{id}^{(k)} &= 1 \\ s_{s_0}^{(k)} &= h_1 = A_0 + A_1 + A_2 \\ s_{s_1 s_0}^{(k)} &= h_2 = A_{02} + A_{21} + A_{10} \\ s_{s_2 s_0}^{(k)} &= h_1^2 - h_2 = A_{20} + A_{12} + A_{01} \\ s_{s_2 s_1 s_0}^{(k)} &= h_2 h_1 = h_1 h_2 = A_{021} + A_{010} + A_{102} + A_{121} + A_{202} + A_{210} \\ s_{s_1 s_2 s_0}^{(k)} &= h_1^3 - h_2 h_1 = A_{120} + A_{010} + A_{201} + A_{121} + A_{202} + A_{012} \end{aligned}$$

3.16. **Proposition.** *We observe the following.*

(a) $s_w^{(k)}$ *has a unique Grassmannian term* A_w.

(b) Λ_n *and* Λ^n *are dual Hopf-algebras under the Hall-inner product. Their Hopf algebra structure is inherited from the Hopf algebra structure of* Λ *with Hopf structure*

$$\Delta_\Lambda(h_i(x)) = \sum_{j\leq i} h_j(x)\otimes h_i(y) \quad \epsilon(f(x)) = f(0) \quad c(h_i(x)) = (-1)^i e_i(x)$$

(c) ψ *is an isomorphism of Hopf algebras. Thus,* $h_0 = A_{id}$ *is the unit,* $\epsilon(b)$ *is the coefficient of* h_0 *when* b *is written as a polynomial in the* h_i*'s, and*

$$\Delta_{\mathbb{B}}(h_i) = \sum_{j\leq i} h_j \otimes h_{i-j}\,.$$

3.17. **Theorem** ([Lam08, Theorem 7.4])**.** $\mathbb{B}$ *and* $\mathbb{B}'$ *are identical as subalgebras of* $\mathbb{A}_0$. *Furthermore, the two Hopf structures agree and we have for each* $w\in W^0$,

$$\phi_0(j(\sigma_{(w)})) = s_w^{(k)}\,.$$

Sketch of subalgebra equality. We will grant that $\mathbb{B}\subseteq\mathbb{B}'$ (for the non-trivial proof, see [Lam08, Subsection 7.1]). By Proposition 3.16(a), $s_w^{(k)}\in\mathbb{B}$ has a unique Grassmannain term A_w and by Proposition 3.8(b), $\phi_0(j(\sigma_{(w)}))$ is the unique element in $\mathbb{B}'$ with unique Grassmannian term A_w. Thus, $\phi_0(j(\sigma_{(w)})) = s_w^{(k)}$ and so the $s_w^{(k)}$'s span $\mathbb{B}'$. Thus, $\mathbb{B}=\mathbb{B}'$. □

4. k-Schur functions as Schubert representatives

4.1. **Theorem.** [Lam08, Theorem 7.1]

(a) *The map* $\theta\colon H_*(\mathrm{Gr})\to\Lambda_n$ *given by*

$$\theta(\sigma_w) = s_w^{(k)}(x)$$

is an isomorphism of Hopf algebras.

(b) *The map* $\theta'\colon H^*(\mathrm{Gr})\to\Lambda^n$ *given by*

$$\theta'(\sigma^w) = \tilde{F}_w(x)$$

is an isomorphism of Hopf algebras.

Proof. Part (a) follows by compositing the following chain of isomorphisms

$$H_*(\mathrm{Gr}) \overset{\sim}{\to} \mathbb{B}' \overset{3.17}{=} \mathbb{B} \overset{\psi^{-1}}{\to} \Lambda_n$$

and the statement or part (b) follows by duality. □

[Lam08] notes some interesting consequences of this theorem.

- This identification gives a topological origin to the Hopf algebra structure of Λ. In particular, while the product structure can be given a topological description via the classical identification with the cohomology of the Grassmannian, the coproduct structure has no prior interpretation. In particular, the Hall inner product can be interpreted as a pairing between homology and cohomology.
- The commutativity of $\mathbb{B} = \mathbb{B}'$ gives topological meaning to the symmetry of the affine Stanley symmetric functions (which include usual Stanley symmetric functions as a special case) via the commutativity of $H_*(\mathrm{Gr})$, which follows from Gr being a double loop space.

5. Combinatorics of k-Schur functions

Historically, k-Schur functions were first introduced for the study of Macdonald polynomials in [LLM03]. From the symmetric function perspective, k-Schur functions have an extra parameter t and were constructed to satisfy the following conjecture.

5.1. **Conjecture** ([LLM03])**.** *For μ a partition let $H_\mu(x;q,t)$ be the "modified" Macdonaly polynomials. Then, for any $k \geq \mu_1$,*

$$H_\mu(x;q,t) = \sum_\nu K^{(k)}_{\nu\mu}(q,t) s^{(k)}_\nu(x;t) \qquad s^{(k)}_\nu = \sum_\lambda \pi^{(k)}_{\lambda\nu}(t) s_\lambda(x)$$

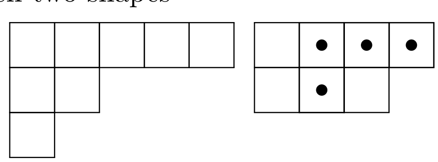
satisfy $K^{(k)}_{\nu\mu}(q,t) \in \mathbb{N}[q,t]$ and $\pi^{(k)}_{\lambda\nu}(t) \in \mathbb{N}[t]$.

Note that [BMPS19] established that $\pi^{(k)}_{\lambda\nu} \in \mathbb{N}[t]$, thus establishing one part of the conjecture. In this presentation, we will specialize $t = 1$, thereby recovering the k-Schur functions from the previous sections. The following propositions and definitions come from [LM03, LM07], but our treatment is a summary of [LLM$^+$14].

5.2. **Definition.** We define the following types of partitions.

(a) A k-bounded partition λ is a partition with $\lambda_1 \leq k$.
(b) An r-core is a partition with where none of its cells have a hook-length equal to r.

5.3. **Example.** Given two shapes

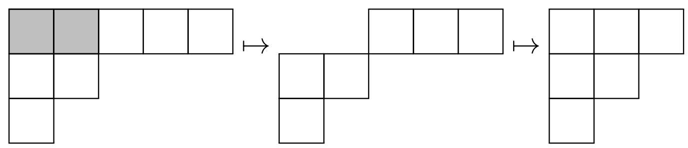


the left shape is a 4-core but the right shape is not. A hook of size 4 is given by the dotted cells.

5.4. **Proposition.** *There is a bijection $\mathfrak{p}$ between the set of $(k+1)$-cores κ and k-bounded partitions given by deleting all cells of hook length greater than $k+1$ in κ and left justifying the remaining cells.*

Such a bijection is best understood via an example.

5.5. **Example.** For $k+1=4$, highlight the cells of hook length ≤ 3 and remove the remaining ones. Then, left justify.



Now, recall the following.

5.6. **Definition.** The *affine symmetric group* $\tilde{S}_n$ is given by the generators $\{s_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ satisfying the relations

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \quad \text{for } i-j \not\equiv 0, 1, n-1 \mod n$$

with all indices considered modulo n.

5.7. **Proposition.** *The Grassmannian elements of $\tilde{S}_n$ are the minimal length coset representatives of $\tilde{S}_n/S_n$ for $S_n = \langle s_1, \ldots, s_n \rangle$ and are characterized by having all reduced words beginning with s_0.*

5.8. **Definition.**
(a) Given a diagram representing a partition, say $\mathrm{dg}(\mu)$, the *content* of a cell $c=(i,j)$ is given by $j-i$.
(b) For a k+1-core κ, the *residue* of a cell (i,j) is given by $j-i \mod k+1$.
(c) We say a cell $c=(i,j)$ is an *addable corner* of a partition μ if $\mathrm{dg}(\mu) \cup \{c\}$ is a diagram for a partition.
(d) We say a cell $c=(i,j)$ is a *removable corner* of a partition μ if $\mathrm{dg}(\mu) \setminus \{c\}$ is a diagram for a partition.
(e) For a $(k+1)$-core, we call an addable (resp. removable) corner of residue r and *addable (resp. removable) r-corner.*

5.9. **Example.** Consider the following 4-core with labelled residues.

$$\kappa = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 0 \\ \hline 3 & 0 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$$

κ has addable corners of residues 1 and 3 and removable corners of residues 0 and 2.

Then, we define the following action of $\tilde{S}_{k+1}$ on $k+1$-cores.

5.10. **Definition.** Given $s_r \in \tilde{S}_{k+1}$ and $k+1$-core κ, we have

$$s_r \cdot \kappa = \begin{cases} \kappa + \text{all its addable } r\text{-corners} & \text{if } \kappa \text{ has at least one addable } r\text{-corner} \\ \kappa - \text{all its removable } r\text{-corners} & \text{if } \kappa \text{ has at least one removable } r\text{-corner} \\ \kappa & \text{else.} \end{cases}$$

5.11. **Proposition.** *There is a bijection* $\mathfrak{t}$ *between* $\tilde{S}^0_{k+1}$ *and* $k+1$*-cores given by sending*

$$w \mapsto s_{i_\ell} \cdots s_{i_1} \cdot \varnothing$$

for $s_{i_\ell} \cdots s_{i_1}$ *a reduced word of* w*.*

5.12. **Definition.** The k-Schur functions $s^{(k)}_\lambda(x)$ are the unique basis of $\Lambda_{k+1} = \mathbb{Z}[h_1, \ldots, h_k]$ satisfying the following Pieri rule: for $0 \leq r \leq k$ and w_λ such that $\mathfrak{p}(\mathfrak{t}(w_\lambda)) = \lambda$

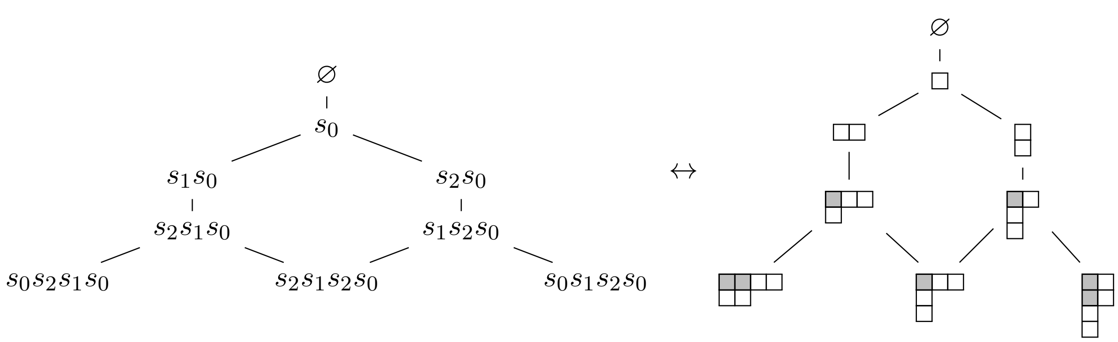
$$h_r s^{(k)}_\lambda(x) = \sum_{\substack{u \in \tilde{S}_{k+1} \\ u \text{ cyclically decreasing} \\ \ell(u)=r \\ \ell(uw_\lambda)=\ell(w_\lambda)+r}} s^{(k)}_{\mathfrak{p}(\mathfrak{t}(uw_\lambda))}(x)$$

This definition is actually equivalent to Definition 3.13(c), but is not very combinatorial, nor very explicit. However, we can work through our bijections to make it more combinatorial. To do this, we will define the notion of "weak order" and "weak tableaux" on cores.

5.13. **Definition.** Consider the affine symmetric group $\tilde{S}_{k+1}$.

(a) The *(left) weak order* on $\tilde{S}_{k+1}$ is given by saying $w \leq v$ if there exists some $u \in \tilde{S}_{k+1}$ such that $uw = v$ and $\ell(u) + \ell(w) = \ell(v)$.
(b) We says w is *covered by* v, denoted $w \lessdot v$, if there is some $s_i \in \tilde{S}_{k+1}$ such that $s_i w = v$ and $\ell(v) = \ell(w) + 1$.
(c) Given two $k+1$-cores κ and τ, we say $\kappa \lessdot \tau$ if $\mathfrak{t}^{-1}(\kappa) \lessdot \mathfrak{t}^{-1}(\tau)$.

5.14. **Example.** The following is the beginning of the Hasse diagram for the weak order on $\tilde{S}^0_3$ and on the corresponding $k+1$-cores.



5.15. **Proposition.** *Given two* $k+1$*-cores* κ *and* τ *with* $\kappa \subseteq \tau$*, then* $\kappa \lessdot \tau$ *if and only if* $\tau = s_i\kappa$ *for some* $s_i \in \tilde{S}_{k+1}$*.*

Using this perspective, we will build up a kind of semistandard tableau using the notion of horizontal strips. To motivate, we will first describe this notion for partitions and semistandard tableau.

5.16. **Definition.** Given two partitions λ and μ, we say λ and μ differ by a *horizontal strip of size* r if $\mu \setminus \lambda$ consists of exactly r cells lying in distinct columns.

From this notion, one can construct a semistandard tableau from a sequence of partitions differing by horizontal strips. E.g., consider the following

$$\varnothing \subseteq \ydiagram{3} \subseteq \ydiagram{5,2} \subseteq \ydiagram{6,3,1} \leftrightarrow \varnothing \subseteq \ytableaushort{111} \subseteq \ytableaushort{11122,22} \subseteq \ytableaushort{111223,223,3}$$

Thus, the chain above corresponds to

$$\ytableaushort{111223,223,3}$$

5.17. **Definition.** Given two $k+1$-cores κ and τ, we say they differ by a *weak horizontal strip* of size $r \leq k$ if

(a) $\tau \setminus \kappa$ is a horizontal strip of partitions,
(b) $|\mathfrak{p}(\tau)| = |\mathfrak{p}(\kappa)| + r$,
(c) and there are exactly r residues in the set of cells $\tau \setminus \kappa$.

5.18. **Proposition.** *For $0 \leq r \leq k$, we have that*

$$h_r s_\lambda^{(k)}(x) = \sum_{\substack{\tau \text{ a } k+1\text{-core} \\ \tau = \mathfrak{p}^{-1}(\lambda) + \text{a weak horizontal } r\text{-strip}}} s_{\mathfrak{p}(\tau)}^{(k)}(x)\,.$$

5.19. **Definition.** A *weak tableau* of shape κ and composition weight α is a sequence of $k+1$-cores $\varnothing = \kappa^{(0)} \subseteq \kappa^{(1)} \subseteq \cdots \subseteq \kappa^{(\ell(\alpha))} = \kappa$ such that $\kappa^{(i)} \setminus \kappa^{(i-1)}$ is a weak horizontal strip of size α_i.

5.20. **Example.** For $k+1 = 3$, a weak tableau of shape $(4,2)$ and weight $(1,1,1,1)$ is given by

$$\varnothing \subseteq \ydiagram{1} \subseteq \ydiagram{2} \subseteq \ydiagram{3,1} \subseteq \ydiagram{4,2} \leftrightarrow \ytableaushort{1234,34}$$

Furthermore, a weak tableau of the same shape weight weight $(2,2)$ is given by

$$\varnothing \subseteq \ydiagram{2} \subseteq \ydiagram{4,2} \leftrightarrow \ytableaushort{1122,22}$$

From this, we get a combinatorial characterization of the affine Schur functions, also called dual k-Schur functions.

5.21. **Proposition.** *The dual k-Schur function $\tilde{F}_\lambda^{(k)}(x) \in \Lambda^{k+1} = \Lambda/\langle m_\lambda \mid \lambda_1 \geq k+1 \rangle$ is the weight generating function of weak tableau of shape $\mathfrak{p}^{-1}(\lambda)$. More preciesly,*

$$F_\lambda^{(k)}(x) = \sum_\alpha \sum_{\substack{T \text{ a weak tableau of shape } \mathfrak{p}^{-1}(\lambda) \\ \text{and weight } \alpha}} x^\alpha$$

where $x^\alpha = \prod_i x_i^\alpha$.

There exist many other equivalent definitions of k-Schur functions and their duals, including a characterization using the strong Bruhat order on $\tilde{S}_{k+1}$. However [BMPS19] makes use of a completely different definition using Young's "raising operators" to prove the Schur positivity of k-Schur functions with a general parameter t.

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