KAZHDAN-LUSZTIG BASIS FOR HECKE ALGEBRAS A CLASS PRESENTATION FOR QUANTUM GROUPS

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1. Introduction

The Kazhdan-Lusztig basis was introduced in [KL79]. We will define the basis and give a proof of its existence and uniqueness, although we will mainly follow the proof in [Soe97]. Since their introduction, the so-called Kazhdan-Lusztig polynomials, which appear in the definition of the basis, have appeared in many other fields of mathematics. For a more detailed overview of connections, see [Bre03, p 5].

2. Preliminaries

We work with the Hecke algebra, for which we will give two presentations.

2.1. **Definition.** [Hum90, Section 7.4] Let $A = \mathbb{Z}[q, q^{-1}]$. Then, the *Hecke algebra* $\mathcal{H}$ associated to a Weyl group $\mathcal{W}$ has a basis $\{T_w \mid w \in \mathcal{W}\}$ with relations

(a) $T_x T_y = T_{xy}$ if $\ell(x) + \ell(y) = \ell(xy)$ and
(b) $T_s^2 = (q-1)T_s + qT_{id}$ for all simple reflections $s \in \mathcal{W}$.

2.2. **Remark.** We need not restrict $\mathcal{W}$ to be a Weyl group. In full generality, we can replace $\mathcal{W}$ with any Coxeter group.

For our purposes, it will also be convenient to work with the Hecke algebra over an enlarged ring. Let $v := q^{-\frac{1}{2}}$. Then, we have the following.

2.3. **Proposition.** *The Hecke algebra over $\mathbb{Z}[v, v^{-1}]$ is given as the associative algebra with generators $\{H_s\}$ for $H_s = vT_s$ and relations*

(a) $H_s^2 = 1 + (v^{-1} - v)H_s$ and
(b) $H_s H_t \cdots H_s = H_t H_s \cdots H_t$ or $H_s H_t H_s \cdots H_t = H_t H_s H_t \cdots H_s$ if $st \cdots s = ts \cdots t$ or $sts \cdots t = tst \cdots s$, repsectively, for simple reflections $s, t \in \mathcal{W}$

2.4. **Proposition.** *The Hecke algebra over $\mathbb{Z}[v, v^{-1}]$ has a basis given by $\{H_w \mid w \in \mathcal{W}\}$ where $H_w = v^{\ell(w)}T_w$. Furthermore, this basis has relation $H_x H_y = H_{xy}$ if $\ell(x) + \ell(y) = \ell(xy)$.*

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2.5. **Lemma.** *We have* $H_s^{-1} = H_s + (v - v^{-1})$ *and so all the* H_x *basis elements are units in* $\mathcal{H}$*.*

Proof.
$$H_s^2 - (v^{-1} - v)H_s = 1 \Longrightarrow H_s(H_s + (v - v^{-1})) = 1$$
□

2.6. **Lemma.** *For simple reflection* $s \in \mathcal{W}$*, if* $\ell(xs) < \ell(x)$*, then*
$$H_x H_s = H_{xs} + (v^{-1} - v)H_x\,.$$

Proof. We know that
$$H_x = H_{xs}H_s \Longrightarrow H_xH_s = H_{xs}H_s^2 = H_{xs}(1+(v^{-1}-v)H_s) = H_{xs}+(v^{-1}-v)H_x$$
□

3. The Kazhdan-Lusztig Basis

3.1. **Definition.** Recall that $A = \mathbb{Z}[q, q^{-1}]$.

(a) We define the $\mathbb{Z}$-linear map, called the *bar involution*, $\bar{\ }\colon A \to A$ given by sending $q \mapsto q^{-1}$
(b) The Hecke algebra $\mathcal{H}$ admits an extension of the bar involution, say $\iota\colon \mathcal{H} \to \mathcal{H}$, given by
$$\iota(T_w) := T_{w^{-1}}^{-1}$$
for any $w \in \mathcal{W}$. For convenience, we will overload notation and write
$$\overline{T_w} := \iota(T_w)$$

Note that $\iota(H_s) = v^{-1}T_s^{-1} = v^{-1}(v^2T_s - 1 + v^2) = H_s - v^{-1} + v = H_s^{-1}$ and, similarly, $\iota(H_w) = H_{w^{-1}}^{-1}$. Then, we have an ι-invariant of the form
$$C_s := q^{-\frac{1}{2}}T_s - q^{\frac{1}{2}}T_{id} = H_s - v^{-1}H_{id}$$
We can also introduce a similar ι-invariant of the form
$$C_s' := H_s + vH_{id}$$
This justifies why we introduced the H-basis in Proposition 2.4. In [Hum90, p 158], it is noted that it could be tempting to construct further ι-invariants by taking products of these C_s elements. However, if one has a word $sts = tst$ with $s, t \in \mathcal{W}$ both simple reflections and $\ell(sts) = 3 = \ell(tst)$, then one can check that $C_sC_tC_s \neq C_tC_sC_t$. However, if we compute (still assuming $\ell(sts) = 3$)
$$C_sC_tC_s - C_t = q^{-\frac{3}{2}}(T_{sts} - q(T_{st} - T_{ts}) + q^2(1+q^{-1})(T_s+T_t) - q^3(1+2q^{-1})T_{id})$$
we get an ι-invariant expression where the s and t's are interchangeable. Similarly, we can compute
$$C_s'C_t'C_s' - C_s' = H_{sts} + v(H_{ts} + H_{st}) + v^2(H_s + H_t) + v^3H_{id}$$
since $vH_s^2 = H_s - v^2H_s + vH_{id}$ and so $vH_s^2 - C_s' = -v^2H_s$.

This illustrates the problem more generally we wish to solve. For every $w \in \mathcal{W}$, we want to associate an ι-invariant element, C_w, which is a linear combination of T_x for $x \leq w$, thus giving us a basis. In order to follow [Soe97], we will actually produce elements C'_w as a linear combination of H_x's, but the idea remains the same. To do this, we first recall a partial ordering on the Weyl group.

3.2. **Definition.** For $u, v \in \mathcal{W}$, we say $u \leq v$ in the (strong) *Bruhat order* on $\mathcal{W}$ if some substring of some reduced word for v is a reduced word for u.

3.3. **Example.** Let $\mathcal{W} = \mathfrak{S}_3 = \langle s_1, s_2 \rangle$. Then, the Bruhat order is given by the following poset.

$$\begin{tikzcd}
 & s_1s_2s_1 & \\
s_1s_2 \arrow[ur] & & s_2s_1 \arrow[ul] \\
s_1 \arrow[u] \arrow[urr] & & s_2 \arrow[u] \arrow[ull] \\
 & id \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

3.4. **Theorem.** [Soe97, Theorem 2.1] *For each $w \in \mathcal{W}$, there exists a unique element $C'_w \in \mathcal{H}$ having the following properties:*

(a) $\iota(C'_w) = C'_w$

(b) $C'_w \in H_w + \sum_{x<w} v\mathbb{Z}[v]H_x$ *where $x < w$ in the (strong) Bruhat order.*

Then, one may wish to construct

3.5. **Example.** (a) From the above, we already see that if $s \in \mathcal{W}$ is a simple reflection, then it must be that

$$C'_s = H_s + vH_{id}$$

(b) We can compute the basis for $\mathfrak{S}_3 = \langle s_1, s_2 \rangle$ by hand. We know that the simple reflections must be of the form.

$$C'_{s_1} = H_{s_1} + vH_{id}$$
$$C'_{s_2} = H_{s_2} + vH_{id}$$

Then, to form ι-invariants of length 2, we check

$$C'_{s_1}C'_{s_2} = H_{s_1s_2} + v(H_{s_1} + H_{s_2}) + v^2H_{id}$$

is ι-invariant. If we apply ι to this, we get

$$\begin{aligned}
\iota(C'_{s_1}C'_{s_2}) &= H_{s_1s_2} + (v - v^{-1})(H_{s_1} + H_{s_2}) + (v - v^{-1})^2 + v^{-1}(H_{s_1} + H_{s_2} + 2(v - v^{-1})) + v^{-2} \\
&= H_{s_1s_2} + v(H_{s_1} + H_{s_2}) + (v - v^{-1})^2 + 2(1 - v^{-2}) + v^{-2} \\
&= H_{s_1s_2} + v(H_{s_1} + H_{s_2}) + v^2
\end{aligned}$$

So, by uniqueness, it must be $C'_{s_1s_2} = C'_{s_1}C'_{s_2}$. A similar computation gives $C'_{s_2s_1}$. For length 3, we already computed above that

$$C'_{s_1s_2s_1} = C'_{s_1}C'_{s_2}C'_{s_1} - C'_{s_1} = H_{s_1s_2s_1} + v(H_{s_1s_2} + H_{s_2s_1}) + v^2(H_{s_1} + H_{s_2}) + v^3$$

Proof of Theorem 3.4. We have already established the formula for C'_s for s a simple reflection. Now, we compute

$$H_x C'_s = \begin{cases} H_{xs} + vH_x & \text{if } xs > x; \\ H_{xs} + v^{-1}H_x & \text{if } xs < x \end{cases}$$

where the first case is immediate from the definition of the Hecke algebra and the second case is a straightforward application of Lemma 2.6. To show existence, we proceed by induction on the Bruhat order. Certainly, $C'_{id} = H_{id} = 1$. Now, let $x \in \mathcal{W}$ be given and suppose we know C'_y exists for all $y < x$. If $x \neq id$, we can find a simple reflection s such that $xs < x$ and by induction, we get

$$C'_{xs}C'_s = H_x + \sum_{y<x} h_y H_y$$

for some $h_y \in \mathbb{Z}[v]$. Then, we say

$$C'_x = C'_{xs}C'_s - \sum_{y<x} h_y(0) C'_y\,.$$

C'_x is ι-invariant because it is a sum of ι-invariant elements and it lies in $H_x + \sum_{y<x} v\mathbb{Z}[v]H_y$ since, if $C'_y = H_y + \sum_{z<y} h_{z,y}H_x$ for $h_{z,y} \in v\mathbb{Z}[v]$, then

$$C'_x = H_x + \sum_{y<x}\left((h_y - h_y(0))H_y - \sum_{z<y} h_y(0)h_{z,y}H_z\right)\,.$$

For uniqueness, we prove the following.

3.6. **Lemma.** *If $H \in \sum_y v\mathbb{Z}[v]H_y$ is ι-invariant, then $H = 0$.*

We have $H_z \in C'_z + \sum_{y<z} \mathbb{Z}[v, v^{-1}]C'_y$ for the C'_x elements described earlier in the proof by the unitriangularity condition. Now, if $H = \sum_y h_y H_y$ and we choose z maximal such that $h_z \neq 0$, then $\iota(H) = H$ implies that $\overline{h_z} = h_z$. However, this contradicts $h_z \in v\mathbb{Z}[v]$, so it must be that $H = 0$.

Thus, if there were two ι-invariant elements C'_w and D'_w satisfying the hypotheses of Theorem 3.4, then it must be that $C'_w - D'_w \in v\mathbb{Z}[v]$ is ι-invariant, but the lemma shows that $C'_w - D'_w = 0$. Thus, uniqueness is established. □

3.7. **Definition.** For $x, y \in \mathcal{W}$, we define the *Kazhdan-Lusztig polynomials* $h_{y,x} \in \mathbb{Z}[v, v^{-1}]$ by the equality

$$C'_x = \sum_y h_{y,x} H_y$$

3.8. **Remark.** These polynomials are related to the Kazhdan-Lusztig polynomials in [KL79], denoted $P_{y,x}$, by

$$h_{y,x} = v^{\ell(x)-\ell(y)} P_{y,x}$$

3.9. **Proposition.** *Let $\mathcal{W}$ be finite, $w_\circ \in \mathcal{W}$ be the longest element, and $r = \ell(w_\circ)$ its length. Then, we have $C'_{w_\circ} = \sum_{y \in \mathcal{W}} v^{r-\ell(y)} H_y$.*

3.1. **Further Properties of Kazhdan-Lusztig Polynomials.** Since their introduction, the Kazhdan-Lusztig polynomials have been an area of intense research. Now, much more is known than when they were first introduced.

3.10. **Proposition.** [KL80] *For any Weyl group $\mathcal{W}$ and $x, y \in \mathcal{W}$, we have that the coefficients a_i occurring in*

$$P_{y,x}(q) = \sum_i a_i q^i$$

satisfy $a_i \in \mathbb{Z}_{\geq 0}$.

3.11. **Remark.** This has been proved by [EW14] for general Coxeter systems.

In [KL79], the following was conjectured. It was proven in [BB81] and [BK81].

3.12. **Proposition.** *Given a semisimple Lie algebra $\mathfrak{g}$ with Weyl group $\mathcal{W}$, for each $w \in \mathcal{W}$, let M_w be the Verma module with heighest weight $-w(\rho) - \rho$ and let L_w be its unique irreducible quotient. Then, we have the equivalent identities*

(a) $\operatorname{ch} L_w = \sum_{y \leq w} (-1)^{\ell(w)+\ell(y)} P_{y,w}(1) \operatorname{ch} M_y$
(b) $\operatorname{ch} M_w = \sum_{y \leq w} P_{w_\circ w, w_\circ y}(1) \operatorname{ch} L_y$

where $w_\circ$ is the longest element of $\mathcal{W}$.

Finally, there exists a geometric interpretation of the Kazhdan-Lusztig polynomials using perverse sheaves.

3.2. **Historical Note.** Kazhdan and Lusztig were originally interested in using the Kazhdan-Lusztig basis to construct representations of the Hecke algebra, but their significance has extended far beyond this goal. Our expoistion here does not follow [KL79] and our definitions do not match those in [KL79], although it is straightforward to translate between [KL79] and these notes. The proof given for existence and uniqueness here is simpler; notably, this exposition does not include the R-polynomials. Such a proof can be found in [Hum90].

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