

# $K$ -theoretic Catalan functions

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- Schubert calculus
- Catalan functions: a new approach to old problems
- $K$ -theoretic Catalan functions

# Overview of Schubert Calculus Combinatorics

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of subvarieties in a variety  $X$ .

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Special basis of polynomials  $\{f_\lambda\}$  such that  $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

# Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of  $\{f_\lambda\}$  enlightens the geometry (and cohomology).

## Goal

Identify  $\{f_\lambda\}$  in explicit (simple) terms amenable to calculation and proofs.

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of Schubert varieties  $\{X_\lambda\}_{\lambda \subseteq (n^m)}$  in variety  $X = \text{Gr}(m, n)$ .

# Classical Schubert Calculus

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## Representatives

Special basis of Schur polynomials  $\{s_\lambda\}$  such that  $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$  for Littlewood-Richardson coefficients  $c_{\lambda\mu}^\nu$ .

## Example

*Semistandard tableaux*: columns increasing and rows non-decreasing.

5			
3	4		
2	3		
1	2	2	5

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Schur function  $s_\lambda$  is a “weight generating function” of semistandard tableaux:

$\begin{array}{ c c } \hline 2 & \\ \hline 1 & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 3 & \\ \hline 1 & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 3 & \\ \hline 2 & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 2 & \\ \hline 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 3 & \\ \hline 1 & 3 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 3 & \\ \hline 2 & 3 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}$
---	---	---	---	---	---	---	---

$$s_{\square} (x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

# Schur functions $s_\lambda$ (cont.)

## Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\square} s_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} = s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}}$$

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Since  $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \dots, \mu_r)} + \text{lower order terms}$ , subtract to get

$$s_{(\mu_1, \dots, \mu_r)} s_\lambda = \sum c_{\lambda\mu}^\nu s_\nu$$

for well-understood *Littlewood-Richardson coefficients*  $c_{\lambda\mu}^\nu$ .

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$$\mathfrak{S}_{s_i} = x_1 + \cdots + x_i$$

### Open Problem

Structure constants  $\mathfrak{S}_w \mathfrak{S}_u = \sum_v c_{wu}^v \mathfrak{S}_v$  have no tableaux description.

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(Co)homology of flag variety	Schubert polynomials
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- $P$ and $Q$ functions
(Co)homology of affine Grassmannian	(dual) $k$ -Schur functions
$K$ -theory of Grassmannian	Grothendieck polynomials
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And many more!

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$$\begin{aligned}\Phi: QH^*(Fl_{k+1}) &\rightarrow H_*(Gr_{SL_{k+1}})_{loc} \\ \mathfrak{S}_w^Q &\mapsto \frac{s_\lambda^{(k)}}{\prod_{i \in Des(w)} \tau_i}\end{aligned}$$

where  $s_\lambda^{(k)}$  is a  $k$ -Schur symmetric function and  $Gr_{SL_{k+1}}$  is the “affine Grassmannian.”

Upshot

# Peterson Isomorphism

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## Upshot

Computations for (quantum) Schubert polynomials can be moved into symmetric functions.

# $k$ -Schur functions

- $s_{\lambda}^{(k)}$  for  $\lambda_1 \leq k$  a basis for  $\mathbb{Z}[s_1, s_2, \dots, s_k]$  (Lapointe et al., 2003).

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- Branching with positive coefficients (Lam et al., 2010):

$$s_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}^{(2)} = s_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}^{(3)} + \underbrace{s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{smallmatrix}}}_{s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}}^{(3)}}$$

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$$s_{(2)}^{(2)} = s_{(2)}^{(3)} + s_{(1,1)}^{(3)} + s_{(1,1,1)}^{(3)}$$

The diagram shows the branching of the Schur function  $s_{(2)}^{(2)}$  into three Schur functions of degree 3. On the left is a 2x2 square Young diagram labeled  $s_{(2)}^{(2)}$ . This is equal to the sum of three Young diagrams: a 2x2 square labeled  $s_{(2)}^{(3)}$ , a Young diagram with two rows (1,1) labeled  $s_{(1,1)}^{(3)}$ , and a Young diagram with three rows (1,1,1) labeled  $s_{(1,1,1)}^{(3)}$ . Brackets are used to group the three diagrams on the right as a single sum.

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The diagram illustrates the branching of the Schur function  $s_{(2)}^{(2)}$  into three Schur functions  $s_{(2)}^{(3)}$ ,  $s_{(1,1)}^{(3)}$ , and  $s_{(1,1,1)}^{(3)}$ . The left side shows  $s_{(2)}^{(2)}$  as a 2x2 grid. The right side shows the sum of three terms:  $s_{(2)}^{(3)}$  (a 2x2 grid),  $s_{(1,1)}^{(3)}$  (a 1x1 grid above a 1x3 grid), and  $s_{(1,1,1)}^{(3)}$  (a 1x1 grid above a 1x2 grid above a 1x1 grid). Brackets indicate that the first term is  $s_{(2)}^{(3)}$  and the sum of the second and third terms is  $s_{(1,1)}^{(3)}$ .

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- Branching with  $t$  important for Macdonald polynomial positivity.

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- (Lam et al., 2010) gives geometric interpretation,
- but no combinatorial interpretation of branching coefficients.
- Branching with  $t$  important for Macdonald polynomial positivity.
- Many conjecturally equivalent definitions.

- Schubert calculus
- **Catalan functions: a new approach to old problems**
- *K*-theoretic Catalan functions

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- 2 From a new definition, (Blasiak et al., 2019) shows the branching coefficients  $b_{\lambda\mu}$  in the expansion  $s_\lambda^{(k)} = \sum_\mu b_{\lambda\mu} s_\mu^{(k+1)}$  have combinatorial interpretation!

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Key: Catalan functions = large class of symmetric functions.

- Raising operators

# Ingredients for Catalan functions

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- Symmetric functions indexed by integer vectors

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- Symmetric functions indexed by integer vectors
- Root ideals

# Raising Operators on Symmetric Functions

- Raising operators  $R_{i,j}$  act on diagrams

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- For  $h_\lambda = s_{\lambda_1} \cdots s_{\lambda_r}$ , we have the *Jacobi-Trudi identity*

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$$s_{22} = (1 - R_{12}) h_{22} = h_{22} - h_{31}$$

$$s_{211} = (1 - R_{12})(1 - R_{23})(1 - R_{13}) h_{211}$$

$$= h_{211} - h_{301} - h_{220} - \color{red}{h_{310}} + \color{red}{h_{310}} + \underbrace{h_{32-1}}_{=0} + h_{400} - \underbrace{h_{41-1}}_{=0}$$

some terms cancel

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Upside: gives definition for Schur function indexed by any integer vector  $\alpha \in \mathbb{Z}^\ell$ . Straightening:

$$s_\alpha = \prod_{i < j} (1 - R_{ij}) h_\alpha = \begin{cases} \pm s_\lambda & \text{for a partition } \lambda \\ 0 & \end{cases}$$

# Raising Operators on Symmetric Functions

Upside: gives definition for Schur function indexed by any integer vector  $\alpha \in \mathbb{Z}^\ell$ . Straightening:

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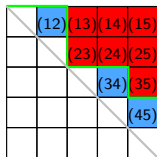
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$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1, \ell], |S|=r} s_{\lambda - \epsilon_S}$$

$$s_{1^2}^\perp s_{333} = s_{322} + s_{232} + s_{223}$$

# Root Ideals

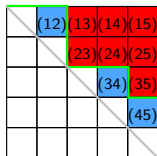
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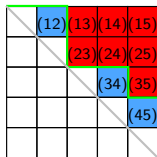
Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For  $\Psi$  and  $\gamma \in \mathbb{Z}^{\ell}$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_{\ell}^{+} \setminus \Psi} (1 - R_{ij}) h_{\gamma}(x)$$

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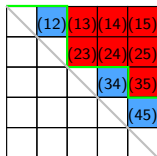
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## Intuition

Catalan functions interpolate between  $h_\lambda$  and  $s_\lambda$ .

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## Theorem (Blasiak et al., 2020)

For  $\Psi$  any root ideal and  $\lambda$  a partition,  $H(\Psi; \lambda)$  is Schur positive!

## $k$ -Schur root ideal for $\lambda$

$$\begin{aligned}\psi &= \Delta^k(\lambda) = \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

← row  $i$  has  $4 - \lambda_i$  non-roots

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$k$ -Schur is a Catalan function (Blasiak et al., 2019).

For partition  $\lambda$  with  $\lambda_1 \leq k$ ,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda).$$

## Key ingredient of branching proof

Dual vertical Pieri rule:  $s_{1^r}^\perp s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$  for  $\langle s_{1^r}^\perp f, g \rangle = \langle f, s_{1^r} g \rangle$ .

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3					
	3				
		2			
			2		
				1	
					1

$$\Delta^5(4, 4, 3, 3, 2, 2) =$$

4					
	4				
		3			
			3		
				2	
					2

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4					
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		3			
			3		
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Pieri:

$$s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda+1^\ell, \mu} s_\mu^{(k+1)}$$

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Branching is a special case of Pieri:

$$s_\lambda^{(k)} = s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda+1^\ell, \mu} s_\mu^{(k+1)}$$

- Schubert calculus
- Catalan functions: a new approach to old problems
- ***K*-theoretic Catalan functions**

# Dual Grothendieck polynomials

- Inhomogeneous basis:  $g_\lambda = s_\lambda +$  lower degree terms.

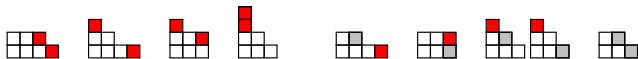
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Add (addable) or mark (removable) in any combination of  $r$  boxes, but only once per row.



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- $g_\lambda = \prod_{i < j} (1 - R_{ij}) k_\lambda$  for  $k_\lambda$  and inhomogeneous analogue of  $h_\lambda$ .
- Dual to Grothendieck polynomials  $G_\lambda$ : Schubert representatives for  $K^*(Gr(m, n))$

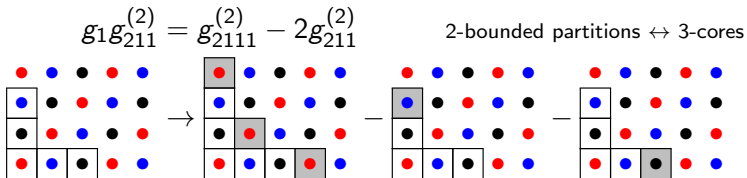
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$$g_{1\mathcal{G}_{211}}^{(2)} = g_{2111}^{(2)} - 2g_{211}^{(2)} \quad \text{2-bounded partitions} \leftrightarrow \text{3-cores}$$

- Conjecture:  $g_\lambda^{(k)}$  have positive branching into  $g_\mu^{(k+1)}$  (Lam et al., 2010; Morse, 2011).

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2-bounded partitions  $\leftrightarrow$  3-cores

The diagram illustrates the Pieri rule for  $K$ - $k$ -Schur functions. It shows the product of a 1-strip and a 211-strip (with 2-bounded partitions shaded) resulting in a 2111-strip minus two 211-strips (with 3-cores shaded).

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## Problem

No direct formula for  $g_\lambda^{(k)}$

## Solution

Find a formula for  $g_{\lambda}^{(k)}$  analogous to raising operator formula for  $s_{\lambda}^{(k)}$ .



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Requires an inhomogeneous refinement of Catalan functions.

# An Extra Ingredient: Lowering Operators

Lowering Operators  $L_j(f_\lambda) = f_{\lambda - \epsilon_j}$

$$L_3 \left( \begin{array}{|c|c|c|c|} \hline \color{red}\square & & & \\ \hline \square & & & \\ \hline \square & \square & \square & \square \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline \square & & \\ \hline \square & \square & \square \\ \hline \end{array}, \quad L_1 \left( \begin{array}{|c|c|c|c|} \hline \square & & & \\ \hline \square & & & \\ \hline \square & \square & \square & \color{red}\square \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline \square & \\ \hline \square & \square \\ \hline \end{array}$$

## $K$ -theoretic Catalan function

Let  $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$  be order ideals of positive roots and  $\gamma \in \mathbb{Z}^\ell$ , then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

# Affine $K$ -Theory Representatives with Raising Operators

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## Example

non-roots of  $\Psi$ , roots of  $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$\begin{aligned} K(\Psi; \mathcal{L}; 54332) \\ = (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12}) (1 - R_{34}) (1 - R_{45}) k_{54332} \end{aligned}$$

Answer (Blasiak-Morse-S., 2020)

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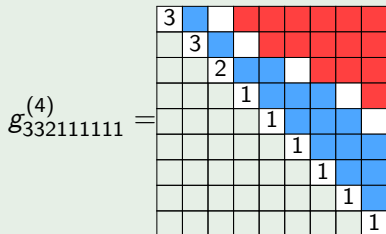
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Example



$$\Delta_9^+ / \Delta^4(332111111), \Delta^5(332111111)$$

# Pieri Rule Illustrated (Recurrences)

A “graphical calculus.”

$$g_1 g_{211}^{(2)}$$



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$$g_1 g_{211}^{(2)}$$

$$= \begin{array}{|c|c|c|c|c|c|} \hline 2 & & & & & \\ \hline & 1 & & & & \\ \hline & & 1 & & & \\ \hline & & & 0 & & \\ \hline & & & & 0 & \\ \hline & & & & & 0 \\ \hline & & & & & 1 \\ \hline \end{array}$$

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$$g_1 g_{211}^{(2)}$$

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

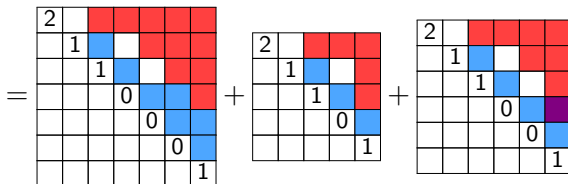
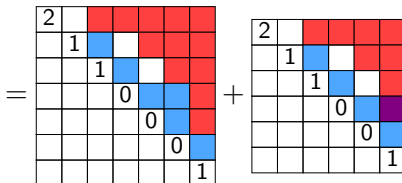
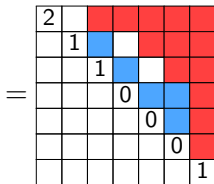
$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

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A “graphical calculus.”

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# Pieri Rule Illustrated (Straightening)

$$g_1 g_{211}^{(2)} =$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			1			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			1			
				0		
					0	
						1

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2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$=$$

2			
	1		
		1	
			1

$$-$$

2		
	1	
		1

$$-$$

2		
	1	
		1

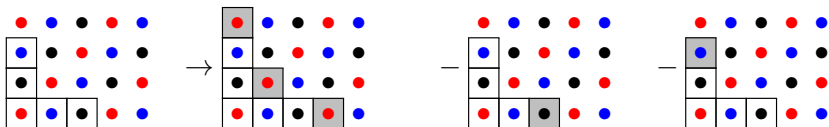
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$$\begin{aligned}
 g_1 g_{211}^{(2)} &= \begin{array}{|c|c|c|c|c|c|} \hline 2 & & & & & \\ \hline & 1 & & & & \\ \hline & & 1 & & & \\ \hline & & & 0 & & \\ \hline & & & & 0 & \\ \hline & & & & & 0 \\ \hline & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|} \hline 2 & & & & & \\ \hline & 1 & & & & \\ \hline & & 1 & & & \\ \hline & & & 0 & & \\ \hline & & & & 0 & \\ \hline & & & & & 0 \\ \hline & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|} \hline 2 & & & & & \\ \hline & 1 & & & & \\ \hline & & 1 & & & \\ \hline & & & 0 & & \\ \hline & & & & 0 & \\ \hline & & & & & 0 \\ \hline & & & & & 1 \\ \hline \end{array} \\
 &= \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline 2 & & \\ \hline & 1 & \\ \hline & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline 2 & & \\ \hline & 1 & \\ \hline & & 1 \\ \hline \end{array} \\
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3-core perspective:



Theorem (Blasiak-Morse-S., 2020)



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The branching coefficients in

$$g_\lambda^{(k)} = \sum_{\mu} a_{\lambda\mu} g_\mu^{(k+1)}$$

satisfy  $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ .

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satisfies  $\tilde{g}_w = g_\lambda^{(k)} + \sum_{\mu} a_{\lambda\mu} g_\mu^{(k)}$  such that  $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ .

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# Closed $K$ - $k$ -Schur functions

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- $K(\Psi; RC^a(\Psi); \lambda) = \sum_{\mu} b_{\mu} s_{\mu}$  satisfies  $b_{\mu} \in \mathbb{Z}_{\geq 0}$ .

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## Thank you!

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