A raising operator formula for Macdonald polynomials via LLT polynomials in the elliptic Hall algebra

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Loyola University Chicago TACO Seminar

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Glad to be back

Graduation May 2015

Outline

1. **Background on symmetric functions and Macdonald polynomials**
2. A new formula for Macdonald polynomials
3. LLT polynomials in the elliptic Hall algebra

Symmetric Group

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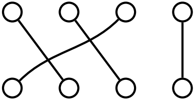

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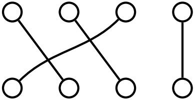

- For $f \in \mathbb{Q}[x_1, \ldots, x_n]$ multivariate polynomial, $\sigma \in S_n$ acts as $\sigma.f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} (5x_1^2 + 5x_2^2 + 8x_3^2) = 8x_1^2 + 5x_2^2 + 5x_3^2$$

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- E.g. for $n = 3$,

$$\begin{aligned} e_1 &= x_1 + x_2 + x_3 = h_1 \\ e_2 &= x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\ e_3 &= x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \cdots \end{aligned}$$

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- Λ is a $\mathbb{Q}$-algebra.

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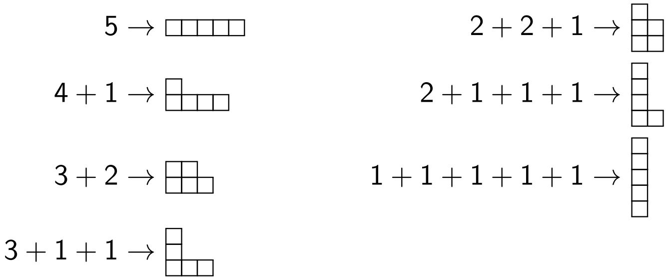

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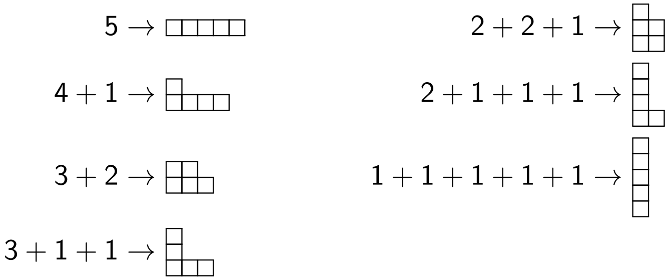

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$\implies$ any basis of degree d symmetric functions can be indexed by partitions of d.

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For $\lambda = (2, 1)$,

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- $\{s_\lambda\}_\lambda$ forms a basis for Λ.

Symmetric functions and Schur functions

- Convention: $h_0 = 1$ and $h_d = 0$ for $d < 0$.
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Precisely, for $\rho = (n-1, n-2, \ldots, 1, 0)$,

$$s_\gamma = \begin{cases} \operatorname{sgn}(\gamma+\rho) s_{\mathsf{sort}(\gamma+\rho)-\rho} & \text{if } \gamma+\rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

- $\mathsf{sort}(\beta)$ = weakly decreasing sequence obtained by sorting β,
- $\operatorname{sgn}(\beta)$ = sign of the shortest permutation taking β to $\mathsf{sort}(\beta)$.

Example: $s_{201} = 0, s_{2\text{-}11} = -s_{200}$.

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Hidden Guide: Schur Positivity

"Naturally occurring" symmetric functions which are non-negative (coefficients in $\mathbb{N}$) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

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Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

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$$\begin{aligned} M &= \mathrm{sp}\left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\qquad x_3 - x_1, x_2 - x_3, 1\} \end{aligned}$$

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$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

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2. How many times does an irreducible S_n-representation occur?

Harmonic polynomials

$$\mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

1. Break M up into irreducible S_n-representations (smallest S_n fixed subspaces).

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

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Remark: $M \cong \mathbb{C}[x_1,x_2,x_3]/(\mathbb{C}[x_1,x_2,x_3]_+^{S_3})$.

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Solution: irreducible S_n-representation of polynomials of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\ydiagram{1,1,1}} + q^2 s_{\ydiagram{1,2}} + q s_{\ydiagram{1,2}} + s_{\ydiagram{3}}$$

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Answer: Hall-Littlewood polynomial $H_{\ydiagram{1,1,1}}(X; q)$.

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- $\tilde{H}_\lambda(X;1,1) = e_1^{|\lambda|}$.
- Does there exist a family of S_n-regular representations whose bigraded Frobenius characteristics equal $\tilde{H}_\lambda(X;q,t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

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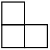
Δ  $= \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3y_2 - y_3x_2 - y_1x_3 + y_1x_2 + y_3x_1 - y_2x_1$

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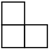

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Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ *has bigraded Frobenius characteristic given by* $\tilde{H}_\lambda(X; q, t)$.

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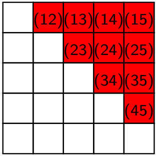
- No combinatorial description of $\tilde{K}_{\lambda\mu}(q,t)$.

Outline

1. Background on symmetric functions and Macdonald polynomials
2. **A new formula for Macdonald polynomials**
3. LLT polynomials in the elliptic Hall algebra

Root ideals

$R_+ = \left\{\alpha_{ij} \mid 1 \leq i < j \leq n\right\}$ denotes the set of positive roots for GL_n, where $\alpha_{ij} = \epsilon_i - \epsilon_j$.



	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

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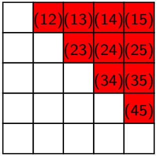
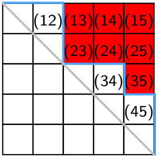


A root ideal $\Psi \subseteq R_+$ is an upper order ideal of positive roots.



$\Psi =$ Roots above Dyck path

Weyl symmetrization

Define the *Weyl symmetrization operator* $\boldsymbol{\sigma} \colon \mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \to \Lambda(X)$ by linearly extending

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$$H(\Phi; \gamma) = \boldsymbol{\sigma} \left(\frac{\boldsymbol{z}^{\gamma}}{\prod_{(i,j) \in \Psi} (1 - t z_i / z_j)} \right)$$

Denominator factors are understood as geometric series $(1 - t z_i / z_j)^{-1} = 1 + t z_i / z_j + t^2 (z_i / z_j)^2 + \cdots$

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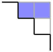

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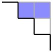

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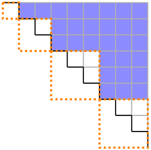
$$\begin{aligned} H(\Psi;\gamma) &= \boldsymbol{\sigma}\left((1 + t\frac{z_1}{z_2} + t^2\frac{z_1^2}{z_2^2} + \cdots)(1 + t\frac{z_1}{z_3} + t^2\frac{z_1^2}{z_3^2} + \cdots)z_1z_2z_3\right) \\ &= s_{111} + t(s_{201} + s_{210}) + t^2(s_{3\text{-}10} + s_{300} + s_{31\text{-}1}) + \cdots \\ &= s_{111} + ts_{210} \end{aligned}$$

A Catalan function for modified Hall-Littlewoods

B_μ = set of roots above block diagonal matrix with block sizes $\mu_{\ell(\mu)}, \ldots, \mu_1$

$B_{3321} =$ 

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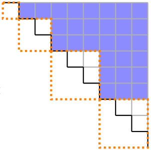


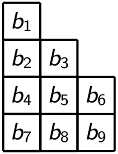
Theorem (Weyman, Shimozono-Weyman)

$$\tilde{H}_\mu(X; 0, t) = \omega\boldsymbol{\sigma}\Big(\frac{z_1 \cdots z_n}{\prod_{\alpha \in B_\mu}(1 - t\boldsymbol{z}^\alpha)}\Big),$$

where $\boldsymbol{z}^\alpha = z_i/z_j$.

$\omega(s_\lambda) = s_{\lambda'}$ for λ' the transpose partition of λ.

Catalan functions for modified Hall-Littlewoods



row reading order

$b_1 \prec b_2 \prec \cdots \prec b_n$

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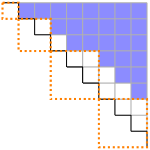

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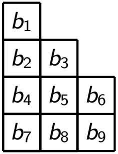

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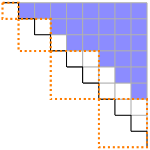

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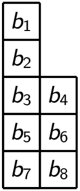
A formula for $\tilde{H}_\mu(X;q,t)$



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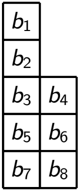

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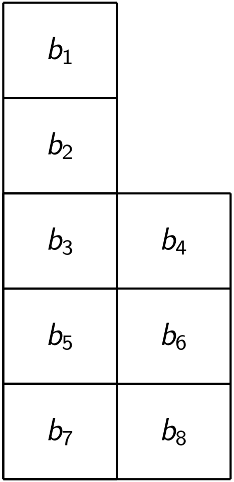
Theorem (Blasiak-Haiman-Morse-Pun-S.)

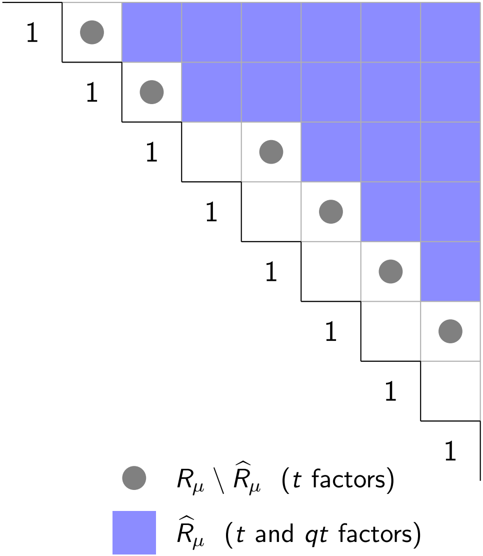
The modified Macdonald polynomial $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$ *is given by*

$$\tilde{H}_\mu = \omega \boldsymbol{\sigma}\left(z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod_{\alpha \in \widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_+} \left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha \in R_\mu} \left(1 - t\boldsymbol{z}^\alpha\right)}\right).$$

Example



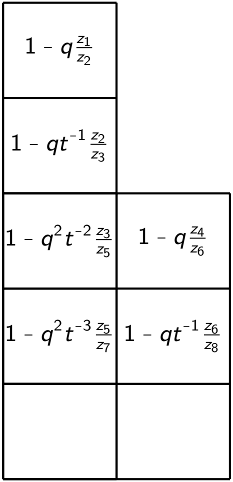
partition $\mu = 22211$



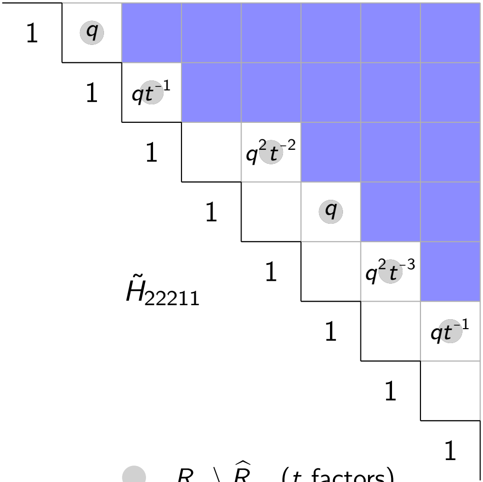
● $R_\mu \setminus \widehat{R}_\mu$ (t factors)

🟦 $\widehat{R}_\mu$ (t and qt factors)

Example



numerator factors $1 - q^{\mathrm{arm}+1} t^{-\mathrm{leg}} z_i/z_j$



● $R_\mu \setminus \widehat{R}_\mu$ (t factors)

🟦 $\widehat{R}_\mu$ (t and qt factors)

$q = t = 1$ specialization

$$
\begin{aligned}
&\omega\boldsymbol{\sigma}\left(z_1\cdots z_n\frac{\displaystyle\prod_{\alpha_{ij}\in R_\mu\setminus\widehat{R}_\mu}\left(1-q^{\mathrm{arm}(b_i)+1}t^{-\mathrm{leg}(b_i)}z_i/z_j\right)\prod_{\alpha\in\widehat{R}_\mu}\left(1-qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+}\left(1-q\boldsymbol{z}^\alpha\right)\prod_{\alpha\in R_\mu}\left(1-t\boldsymbol{z}^\alpha\right)}\right)\\
\overset{q=t=1}{\rightarrow}&\omega\boldsymbol{\sigma}\left(z_1\cdots z_n\frac{\prod_{\alpha\in R_\mu\setminus\widehat{R}_\mu}(1-\boldsymbol{z}^\alpha)\prod_{\alpha\in\widehat{R}_\mu}(1-\boldsymbol{z}^\alpha)}{\prod_{\alpha\in R_+}(1-\boldsymbol{z}^\alpha)\prod_{\alpha\in R_\mu}(1-\boldsymbol{z}^\alpha)}\right)\\
=&\omega\boldsymbol{\sigma}\left(\frac{z_1\cdots z_n}{\prod_{\alpha\in R_+}(1-\boldsymbol{z}^\alpha)}\right)\\
=&\omega h_1^n\\
=&e_1^n
\end{aligned}
$$

$q = 0$ specialization

$$\begin{aligned}
&\omega\boldsymbol{\sigma}\left(z_1\cdots z_n\frac{\displaystyle\prod_{\alpha_{ij}\in R_\mu\setminus\widehat{R}_\mu}\left(1-q^{\mathrm{arm}(b_i)+1}t^{-\mathrm{leg}(b_i)}z_i/z_j\right)\prod_{\alpha\in\widehat{R}_\mu}\left(1-qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+}\left(1-q\boldsymbol{z}^\alpha\right)\prod_{\alpha\in R_\mu}\left(1-t\boldsymbol{z}^\alpha\right)}\right)\\
\overset{q=0}{\to}&\omega\boldsymbol{\sigma}\left(\frac{z_1\cdots z_n}{\prod_{\alpha\in R_\mu}(1-t\boldsymbol{z}^\alpha)}\right)\\
=&\tilde{H}_\mu(X;0,t)
\end{aligned}$$

Proof of formula for $\tilde{H}_\mu$

Definition

∇ is the linear operator on symmetric functions satisfying $\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu$, where $n(\mu) = \sum_i (i-1)\mu_i$.

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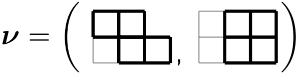
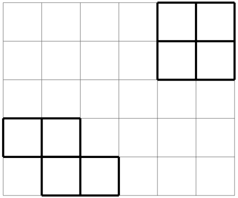
- Apply $\omega\nabla$ to both sides.
- Use Catalan-like ("Catalanimal") formula for $\omega\nabla\mathcal{G}_{\boldsymbol{\nu}}(X;q)$ and collect terms.

LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.





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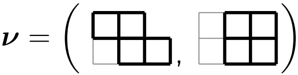

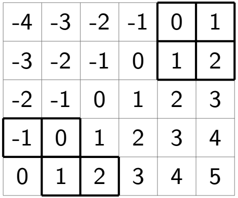

-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

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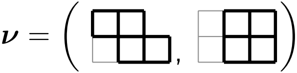

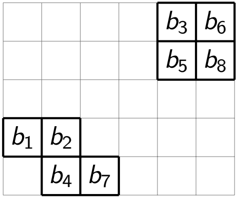

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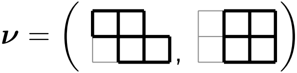

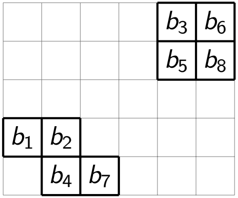

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

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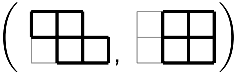

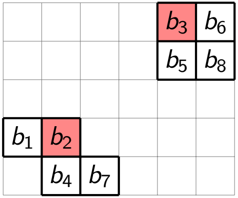

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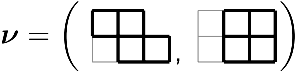

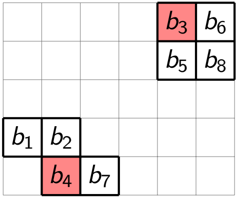

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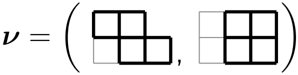

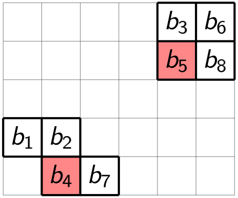

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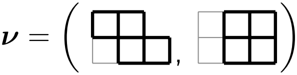

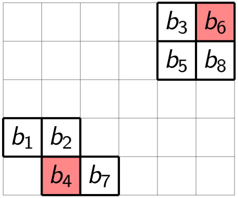

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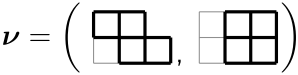

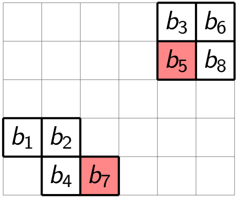

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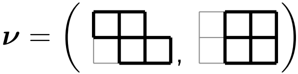

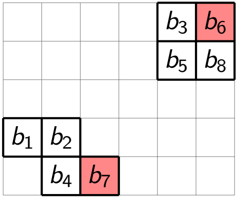

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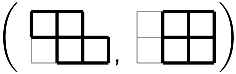

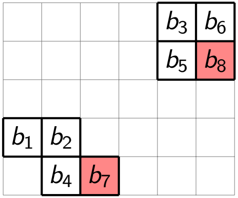

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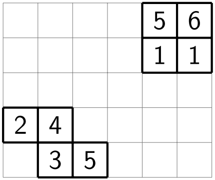
- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T \colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{x}; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

$T \quad =$



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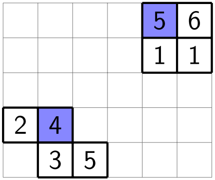

non-inversion

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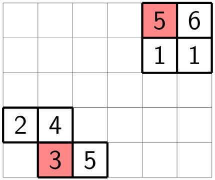

inversion

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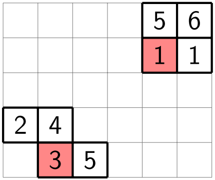

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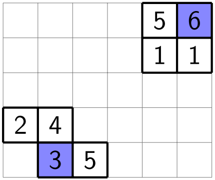
$T \;=$



non-inversion

LLT Polynomials

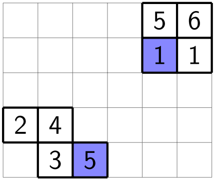
- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{x}; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

$T \quad =$



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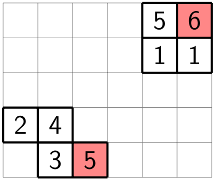

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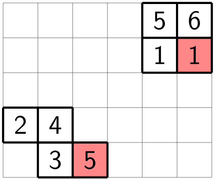

inversion

$\mathrm{inv}(T) = 4, \quad \boldsymbol{x}^T = x_1^2 x_2 x_3 x_4 x_5^2 x_6$

Catalanimals

Definition

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$$H(R_q, R_t, R_{qt}, \lambda) = \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^\lambda \prod_{\alpha \in R_{qt}} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_q} \left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha \in R_t} \left(1 - t\boldsymbol{z}^\alpha\right)} \right).$$

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With $n = 3$,

$$\begin{aligned} H(R_+, R_+, \{\alpha_{13}\}, (111)) &= \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^{111}(1 - qtz_1/z_3)}{\prod_{1 \leq i < j \leq 3}(1 - qz_i/z_j)(1 - tz_i/z_j)}\right) \\ &= s_{111} + (q + t + q^2 + qt + t^2)s_{21} + (qt + q^3 + q^2t + qt^2 + t^3)s_3 \\ &= \omega \nabla e_3. \end{aligned}$$

LLT Catalanimals

For a tuple of skew shapes $\boldsymbol{\nu}$, the *LLT Catalanimal* $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ is determined by

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- λ: fill each diagonal D of $\boldsymbol{\nu}$ with
  $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.
  Listing this filling in reading order gives λ.

LLT Catalanimals

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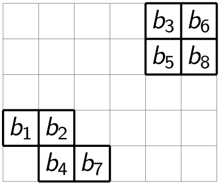
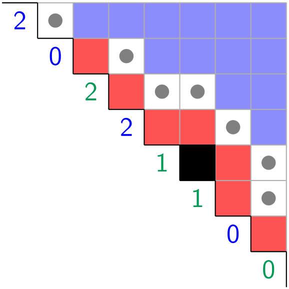


$\boldsymbol{\nu}$



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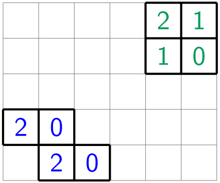
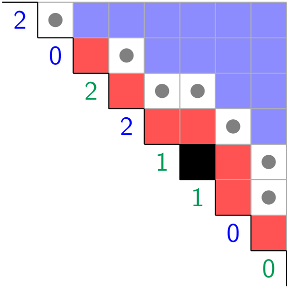


λ, as a filling of $\boldsymbol{\nu}$



LLT Catalanimals

Theorem (Blasiak-Haiman-Morse-Pun-S.)

Let $\boldsymbol{\nu}$ *be a tuple of skew shapes and let* $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ *be the associated LLT Catalanimal. Then*

$$\begin{aligned}\nabla \mathcal{G}_{\boldsymbol{\nu}}(X;q) &= c_{\boldsymbol{\nu}}\, \omega H_{\boldsymbol{\nu}} \\ &= c_{\boldsymbol{\nu}}\, \omega \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^{\lambda} \prod_{\alpha \in R_{qt}} \left(1 - qt\, \boldsymbol{z}^{\alpha}\right)}{\prod_{\alpha \in R_q} \left(1 - q\, \boldsymbol{z}^{\alpha}\right) \prod_{\alpha \in R_t} \left(1 - t\, \boldsymbol{z}^{\alpha}\right)} \right)\end{aligned}$$

for some $c_{\boldsymbol{\nu}} \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$.

Haglund-Haiman-Loehr formula

Theorem (Haglund-Haiman-Loehr, 2005)

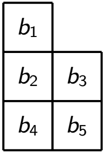
$$\tilde{H}_\mu(X;q,t) = \sum_D \left(\prod_{u \in D} q^{-\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1} \right) \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}(X;q)\,,$$

where

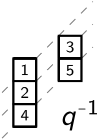
- *the sum runs over all subsets* $D \subseteq \{(i,j) \in \mu \mid j > 1\}$*, and*
- $\boldsymbol{\nu}(\mu, D) = (\nu^{(1)}, \ldots, \nu^{(k)})$ *where* $k = \mu_1$ *is the number of columns of* μ*, and* $\nu^{(i)}$ *is a ribbon of size* μ_i^**, i.e., box contents* $\{-1, -2, \ldots, -\mu_i^*\}$*, and descent set* $Des(\nu^{(i)}) = \{-j \mid (i,j) \in D\}$*.*

Haglund-Haiman-Loehr formula example

$$\tilde{H}_\mu(X;q,t) = \sum_D \left(\prod_{u\in D} q^{-\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1}\right) \mathcal{G}_{\nu(\mu,D)}(X;q)$$

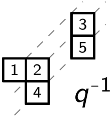


μ



$q^{-1}t^4$

$D = \{b_1, b_2, b_3\}$



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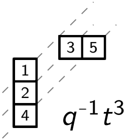

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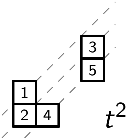
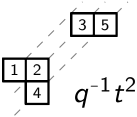


t^2

$D = \{b_1, b_3\}$



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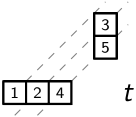

t

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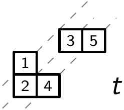
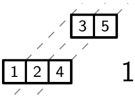


t

$D = \{b_1\}$



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$D = \varnothing$

Putting it all together

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}$ and apply $\omega\nabla$.

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- By construction, all the LLT Catalanimals $H_{\boldsymbol{\nu}(\mu,D)}$ appearing on the RHS will have the same root ideal data (R_q, R_t, R_{qt}).
- Collect terms to get $\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} (1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j)$ factor.

$$\tilde{H}_\mu = \omega \boldsymbol{\sigma} \left(z_1 \cdots z_n \frac{\displaystyle\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod_{\alpha \in \widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_+} \left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha \in R_\mu} \left(1 - t\boldsymbol{z}^\alpha\right)} \right).$$

Outline

1. Background on symmetric functions and Macdonald polynomials
2. A new formula for Macdonald polynomials
3. **LLT polynomials in the elliptic Hall algebra**

Elliptic Hall Algebra

Burban and Schiffmann studied a subalgebra $\mathcal{E}$ of the Hall algebra of coherent sheaves on an elliptic curve over $\mathbb{F}_p$.

The *elliptic Hall algebra* $\mathcal{E}$ is generated by subalgebras $\Lambda(X^{a,b})$ isomorphic to the ring of symmetric functions Λ over $\mathbb{k} = \mathbb{Q}(q, t)$, one for each coprime pair $(a, b) \in \mathbb{Z}^2$, along with an additional central subalgebra.

Shuffle algebra

Define a linear map

$$\sigma_\Gamma \colon \bigoplus_n \mathbb{k}(z_1, \dots, z_n) \to \bigoplus_n \mathbb{k}(z_1, \dots, z_n)^{S_n}$$

whose graded components σ_Γ^n are given by

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The *shuffle algebra* $\mathcal{S}_\Gamma$ is the image of $\bigoplus_n \mathbb{k}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ under the map σ_Γ, equipped with a variant of the concatenation product.

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Nice fact (up to some modifications of definitions)

Some Catalanimals are elements in $\mathcal{S}_\Gamma$. ("Tame Catalanimals")

Shuffle to elliptic Hall isomorphism

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Theorem (Schiffmann-Vasserot)

There is an algebra isomorphism $\psi \colon \mathcal{S}_\Gamma \to \mathcal{E}^+$.

Elliptic Hall algebra action

Schiffmann-Vasserot and Feigin-Tsymbaliuk constructed an action of $\mathcal{E}$ on Λ, where $f(X^{0,1})$ acts by multiplication by $f(X)$.

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Proposition

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$$\nabla f(X^{a,b}) \nabla^{-1} = f(X^{a+b,b}).$$

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$f(X^{1,1}) \cdot 1 = \nabla f(X^{0,1}) \nabla^{-1} \cdot 1 = \nabla f.$

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Theorem (Blasiak-Haiman-Morse-Pun-S.)

Let H be a Catalanimal such that $\psi(H) = f(X^{1,1})$. Then

$$\nabla f = \omega H\,.$$

Shuffle to elliptic Hall summary

$$\begin{tikzcd}
 & \mathcal{E} & \curvearrowright \Lambda & f(X^{1,1}) \cdot 1 = \nabla f \\
\bigoplus_{\substack{a>0 \\ b\in\mathbb{Z} \\ (a,b)=1}} \Lambda(X^{a,b}) \arrow[r, phantom, "\underset{\text{v.sp.}}{\cong}"] & \mathcal{E}^{+} \arrow[u, hook] & & \\
\sigma_\Gamma \left(\bigoplus_n \Bbbk[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]\right) \arrow[r, phantom, "\underset{\text{v.sp.}}{\cong}"] & \mathcal{S}_\Gamma \arrow[u, "\psi", "\cong"'] & \ni H & \text{“tame” Catalanimal}
\end{tikzcd}$$

Theorem (Blasiak-Haiman-Morse-Pun-S.)

$\psi(H) = f(X^{1,1}) \Longrightarrow f(X^{1,1}) \cdot 1 = \nabla f = \omega H.$

Proof of $\nabla \mathcal{G}_{\nu}$ formula

1. LLT Catalanimals $H_{\boldsymbol{\nu}}$ are tame.
2. LLT Catalanimals lie in $\psi^{-1}(\Lambda(X^{1,1}))$.
3. Describe coproduct Δ on $\mathcal{E}$ explicitly on tame Catalanimals and show $\Delta H_{\boldsymbol{\nu}}$ matches $\Delta \mathcal{G}_{\boldsymbol{\nu}}$.
4. Conclude $\psi(H_{\boldsymbol{\nu}}) = c_{\boldsymbol{\nu}}^{-1} \mathcal{G}_{\boldsymbol{\nu}}(X^{1,1}) \in \mathcal{E}$.
5. Apply previous theorem to conclude $\nabla \mathcal{G}_{\boldsymbol{\nu}} = c_{\boldsymbol{\nu}} \omega H_{\boldsymbol{\nu}}$

A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

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$$\tilde{H}_\mu^{(s)} := \omega \boldsymbol{\sigma} \left((z_1 \cdots z_n)^s \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod_{\alpha \in \widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_+} \left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha \in R_\mu} \left(1 - t\boldsymbol{z}^\alpha\right)} \right)$$

Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition μ and positive integer s, the symmetric function $\tilde{H}_\mu^{(s)}$ is Schur positive. That is, the coefficients in

$$\tilde{H}_\mu^{(s)} = \sum_\nu K_{\nu,\mu}^{(s)}(q,t)\, s_\nu(X)$$

satisfy $K_{\nu,\mu}^{(s)}(q,t) \in \mathbb{N}[q,t]$.

Thank you!

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Catalanimals in the shuffle algebra

For $\lambda \in \mathbb{Z}^n$,

$$\begin{aligned}\sigma_\Gamma^n(\boldsymbol{z}^\lambda) &= \sum_{w \in S_n} w\left(\frac{\boldsymbol{z}^\lambda \prod_{\alpha \in R_+}\left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_+}\left(\left(1 - \boldsymbol{z}^{-\alpha}\right)\left(1 - q\boldsymbol{z}^\alpha\right)\left(1 - t\boldsymbol{z}^\alpha\right)\right)}\right) \\ &= H(R_+, R_+, R_+, \lambda) \in \mathcal{S}_\Gamma.\end{aligned}$$

- Technicality: we have redefined $\boldsymbol{\sigma}(\boldsymbol{z}^\gamma) = \sum_{w \in S_n}\left(\frac{\boldsymbol{z}^\gamma}{\prod_{\alpha \in R_+}(1 - \boldsymbol{z}^{-\alpha})}\right) = \chi_\gamma$, the irreducible GL_n character.
- Let pol_X send $\chi_\lambda \mapsto s_\lambda$ if $\lambda_n \geq 0$, otherwise $\chi_\lambda \mapsto 0$.
- The $\boldsymbol{\sigma}$ from before is given by $\boldsymbol{\sigma}_{\mathrm{old}} = \mathrm{pol}_X\, \boldsymbol{\sigma}_{\mathrm{new}}$.

Catalanimals in the Shuffle algebra

$\sigma_\Gamma^n(f)$ can lie in $\mathcal{S}_\Gamma$ even when f is not a Laurent polynomial.

Theorem (Negut)

The following family of Catalanimals lie in the shuffle algebra:

$$\sigma_\Gamma^n\Big(\frac{\boldsymbol{z}^\lambda}{\prod_{i=1}^{n-1}(1-qtz_i/z_{i+1})}\Big) = H(R_+, R_+, R'_+, \lambda) \in \mathcal{S}_\Gamma,$$

where $\quad R'_+ = \{\alpha_{ij} \in R_+ \mid i+1 < j\}.$

The wheel condition

- A symmetric Laurent polynomial $g(\boldsymbol{z})$ satisfies the *wheel condition* if it vanishes whenever any three of the variables z_i, z_j, z_k are in the ratio $(z_i : z_j : z_k) = (1 : q : qt) = (1 : t : qt)$.
- Let $\mathcal{S}_{\check{\Gamma}} \cong \mathcal{S}_{\Gamma}$ for $\check{\Gamma}(z_i, z_j) = (1 - z_i/z_j)(1 - qz_j/z_i)(1 - tz_j/z_i)(1 - qtz_i/z_j)$.

Theorem (Negut)

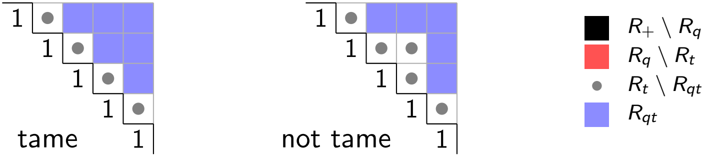
A symmetric Laurent polynomial $g(z_1, \ldots, z_n)$ belongs to $\mathcal{S}_{\check{\Gamma}}$ if and only if it satisfies the wheel condition and vanishes whenever $z_i = z_j$ for $i \neq j$.

The wheel condition and tame Catalanimals

A Catalanimal $H(R_q, R_t, R_{qt}, \lambda)$ is *tame* if

$$R_q + R_t \subseteq R_{qt},$$

where $R_q + R_t = \{\alpha + \beta \mid \alpha \in R_q, \beta \in R_t\}$.



The Catalanimals $H(R_+, R_+, R'_+, \lambda)$ and the LLT Catalanimals are tame.

Using Negut's theorem, we show: Tame Catalanimals belong to the shuffle algebra $\mathcal{S}_\Gamma$.