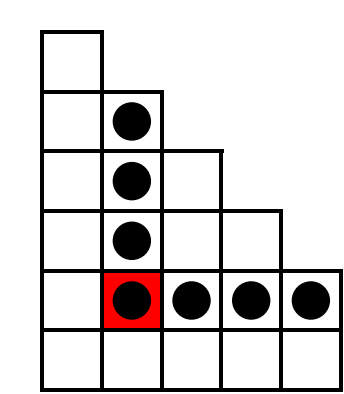


Overview

- Schur functions, s_λ , and Grothendieck polynomials, G_λ , give representatives for cohomology and K -theory of the Grassmannian.
- Pieri rules determine the structure constants of these rings.
- Representatives are known for (co)homology of affine Grassmannian.
- Aim: develop similar picture for affine K -theory.

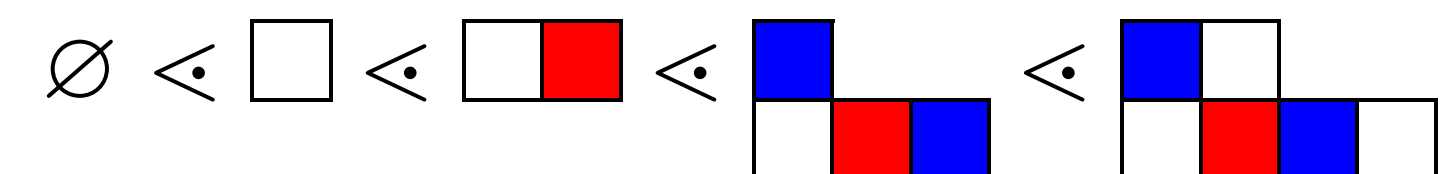
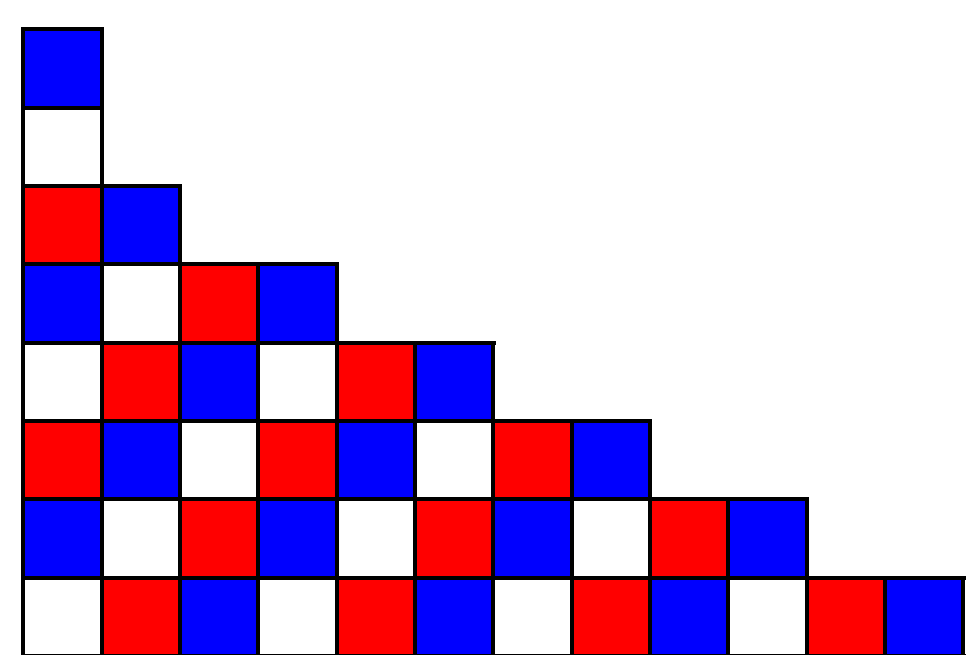
Affine Combinatorics

$w \in \tilde{S}_n \leftrightarrow n$ -cores
 n -core=partition with no cell of hook-length n
 red cell has hook-length 7



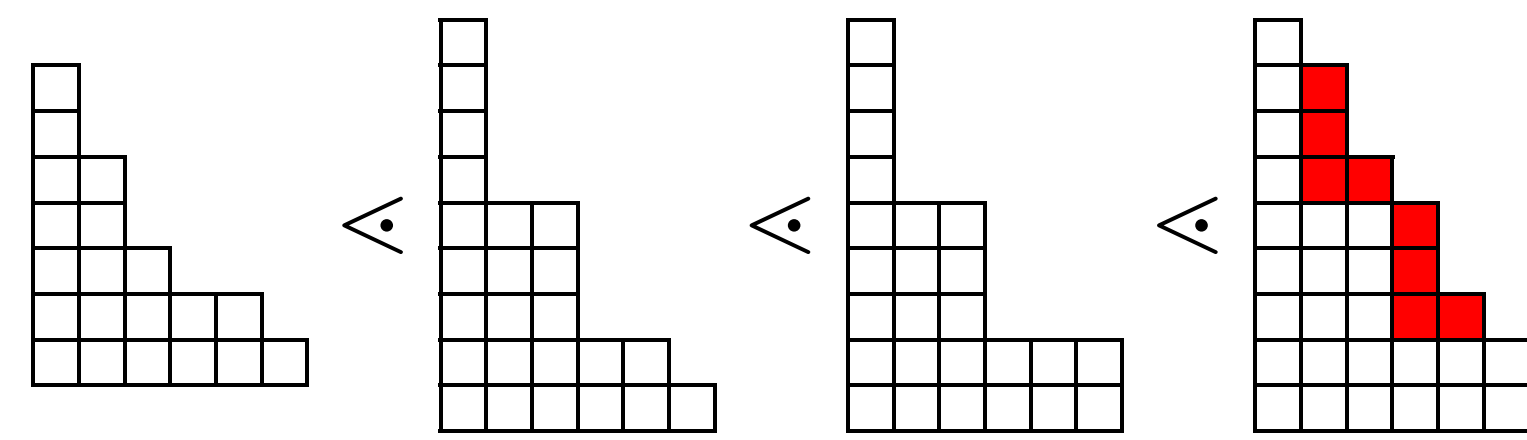
Weak Order

- Covers differ by boxes of same color.

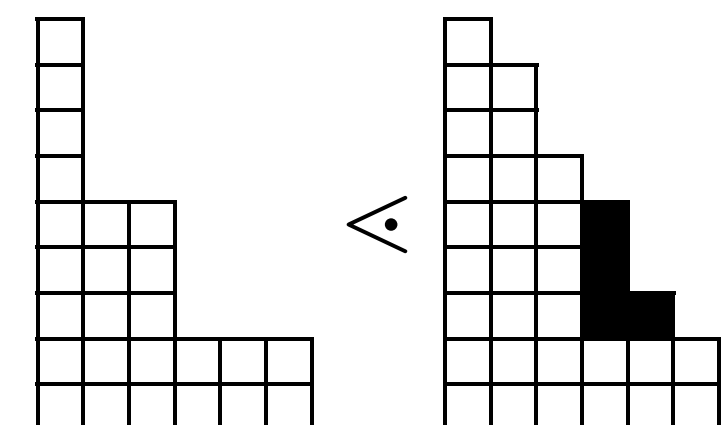


Strong (Bruhat) Order

- Ordered by containment of shapes.
- Covers differ by a ribbon + its copies.



- Marked Cover:** Strong cover with selection of one ribbon



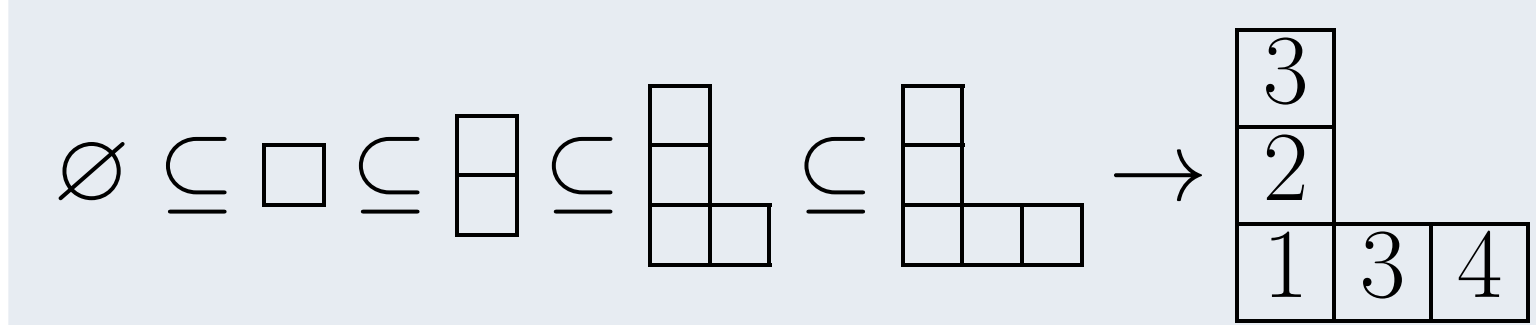
Dual k -Schur Functions

Generating functions of weak tableaux.

$$F_\lambda^{(k)} := \sum x^{\text{weight}(T)}$$

Weak Tableaux:

Maximal chains in the weak order.



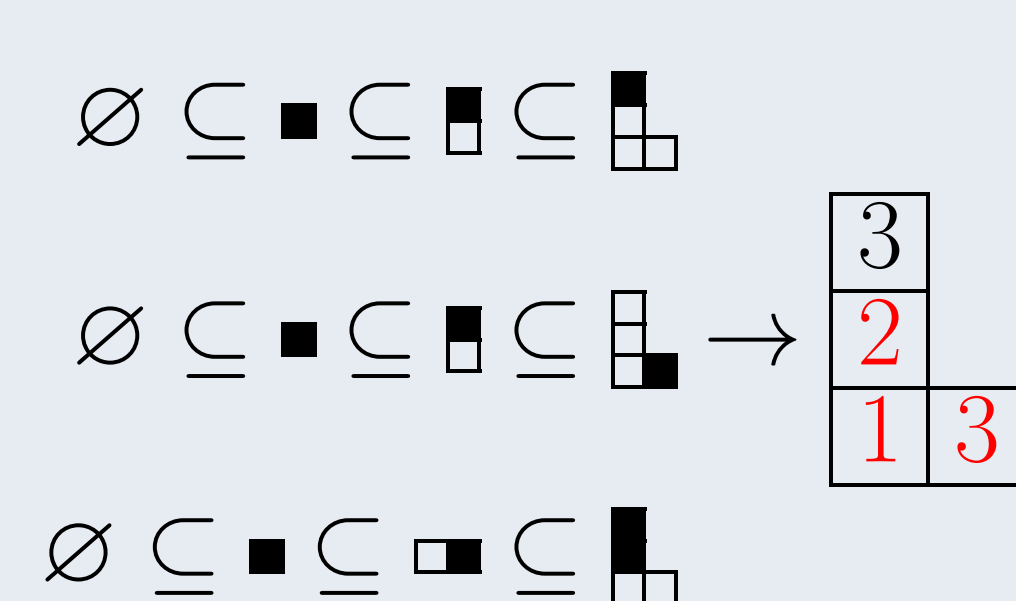
k -Schur Functions

Generating functions of strong tableaux.

$$s_\lambda^{(k)} := \sum x^{\text{weight}(T)}$$

Strong Tableaux:

Maximal strong order chains of marked covers



Pieri Rule:

$$e_r F_\lambda^{(k)} = \sum_{\mu=\lambda+\text{strong marked vertical strip of size } r} F_\mu^{(k)} \iff e_r^\perp s_\mu^{(k)} = \sum_{\lambda=\mu-\text{strong marked vertical strip of size } r} s_\lambda^{(k)}$$

where a **strong vertical strip** is a chain of marked covers with markings proceeding north to south.

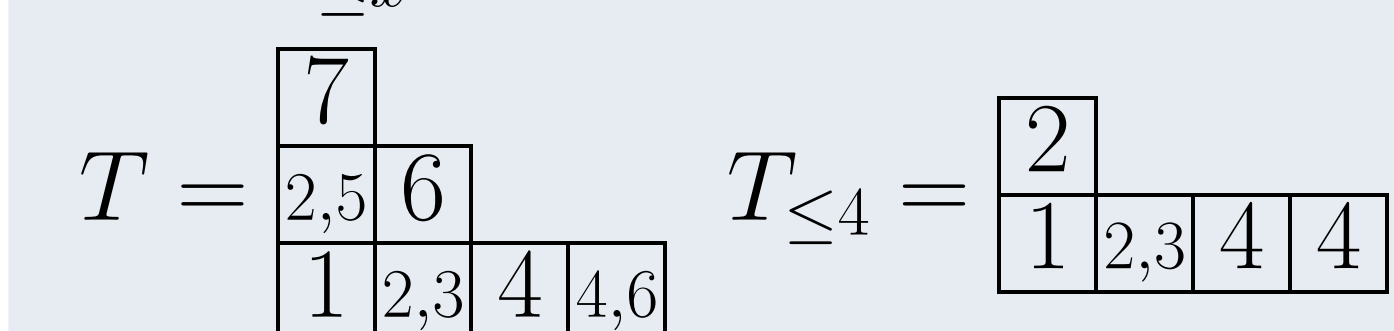
Affine Grothendieck Polynomials

Generating functions of affine SVTs.

$$G_\lambda^{(k)} := \sum (-1)^{|\lambda|} x^{\text{weight}(T)}$$

Affine Set-Valued Tableaux:

Each $T_{\leq x}$ is a $k+1$ -core.



- $G_\lambda^{(k)} = F_\lambda^{(k)} + \text{higher order terms}$
- $G_\lambda^{(k)} = G_\lambda$ for large k .

Dual Affine Grothendieck Polynomials

$g_\lambda^{(k)}$ is dual basis to $G_\lambda^{(k)}$.

- $g_\lambda^{(k)} = s_\lambda^{(k)} + \text{lower order terms}$
- $g_\lambda^{(k)} = g_\lambda$, dual to G_λ , for large k .

Open Problem

Find a direct definition of $g_\lambda^{(k)}$.

Open Problem

Describe the $G_\lambda^{(k)}$ Pieri rule. \iff Describe the $g_\lambda^{(k)}$ dual Pieri rule.

Branching

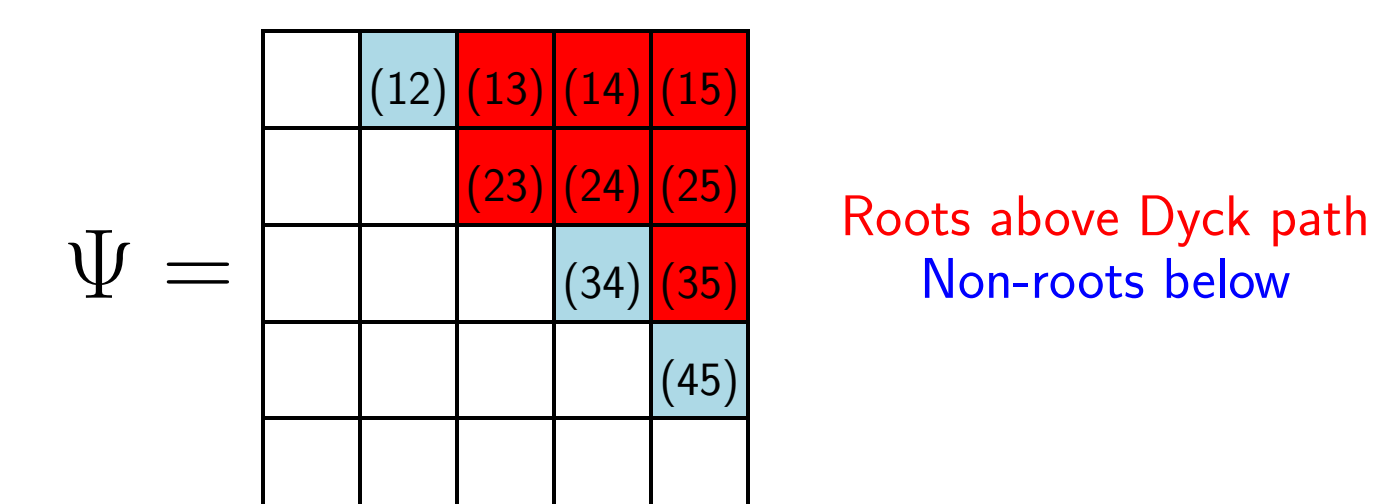
- k -Schur functions are $k+1$ -Schur positive: $s_\lambda^{(k)} = \sum a_{\lambda\mu}^{(k)} s_\mu^{(k+1)}$ with $a_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$.
- Iteration gives Schur positivity of k -Schur functions.
- Conjecture:** $g_\lambda^{(k)}$ is Schur positive.
- Conjecture:** $g_\lambda^{(k)} = \sum_\mu (-1)^{|\lambda|-|\mu|} b_{\lambda\mu}^{(k)} g_\mu^{(k+1)}$ for $b_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$.

Catalan Functions

For $\gamma \in \mathbb{Z}^\ell$,

$$H(\Psi; \gamma) := \prod_{(i,j) \notin \Psi} (1 - R_{ij}) h_\gamma$$

- Raising operators $R_{i,j}(h_\lambda) = h_{\lambda+\epsilon_i-\epsilon_j}$
- Root ideal Ψ : given by Dyck path.

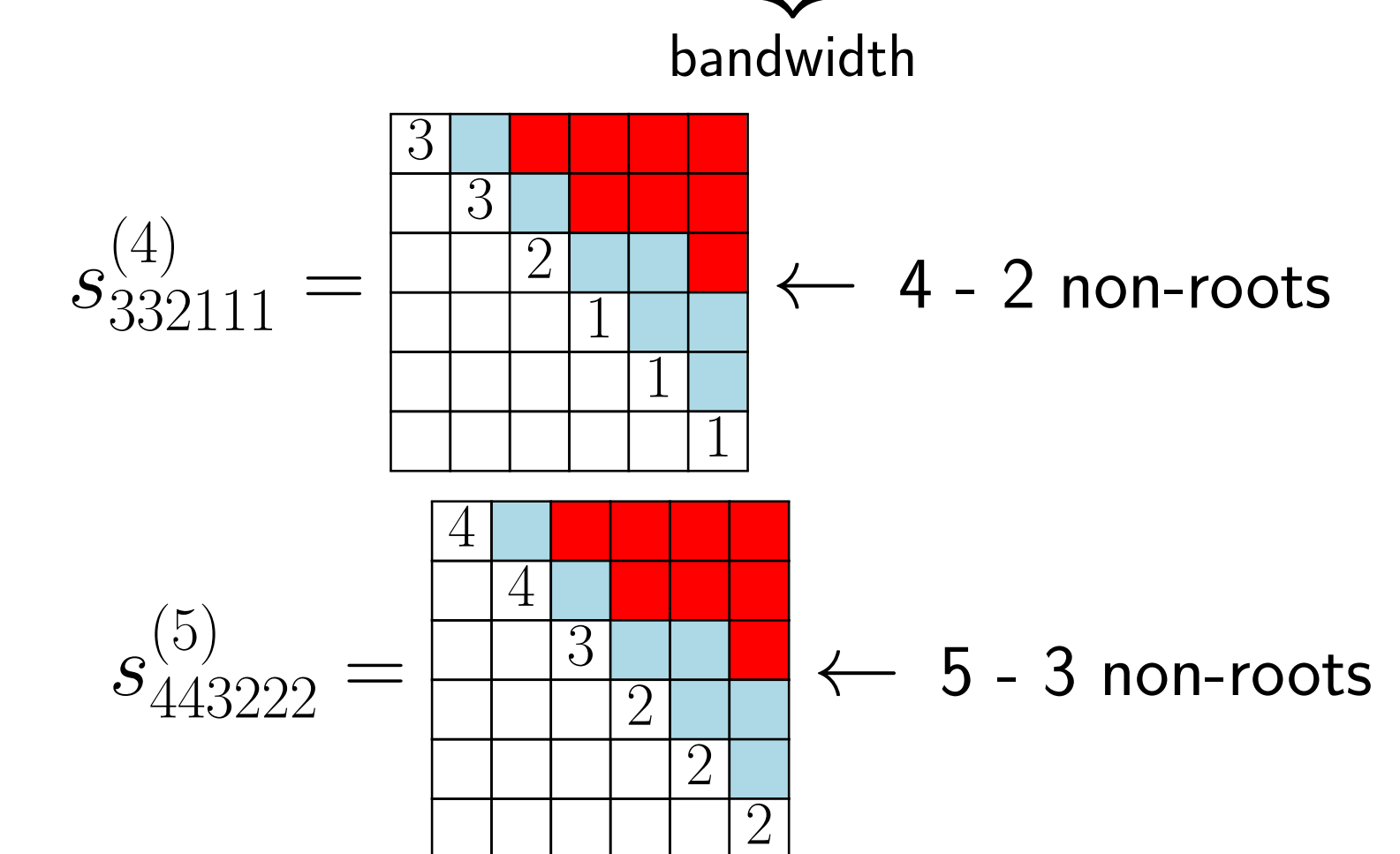


$$\begin{aligned} H(\Psi; 54332) &= (1 - R_{12})(1 - R_{34})(1 - R_{45}) h_{54332} \\ &= h_{54332} - h_{45332} - h_{54422} - h_{54341} \\ &\quad + h_{45422} + h_{45341} + h_{54431} - h_{45431} \end{aligned}$$

- $H(\emptyset; \lambda) = s_\lambda$ (Jacobi-Trudi Identity)

k -Schur Catalans

$s_\lambda^{(k)} = H(\Psi; \lambda)$ for particular Ψ , defined by $\lambda_i + \# \text{non-roots in row } i = k$.



Why use Catalan k -Schurs?

Shift Invariance:

$$e_\ell^\perp s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}$$

Corollary

Dual Pieri rule $\implies s_\lambda^{(k)}$ branching!

$$s_\lambda^{(k)} = e_\ell^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda\mu}^{(k)} s_\mu^{(k+1)}$$

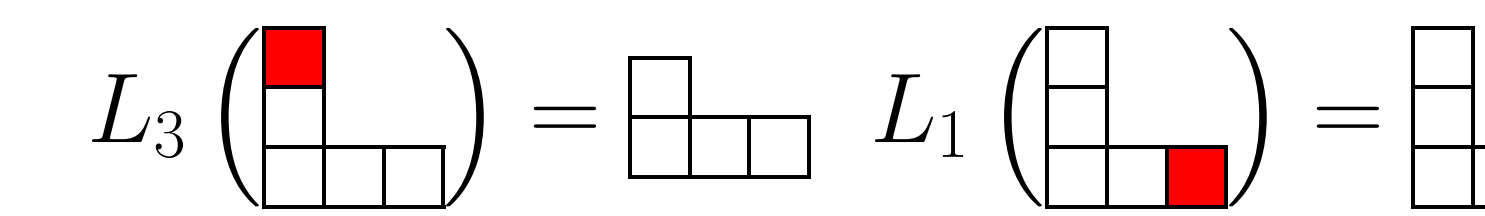
where $a_{\lambda\mu}^{(k)}$ counts strong vertical strips.

K-theoretic Catalan Functions

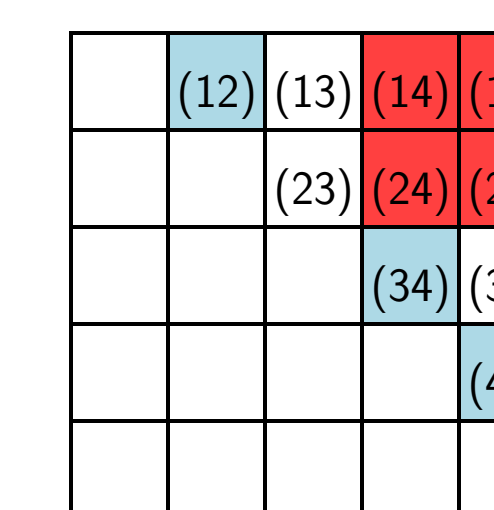
For $\gamma \in \mathbb{Z}^\ell$, root ideals Ψ, \mathcal{L}

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \notin \Psi} (1 - R_{ij}) K h_\gamma$$

- Lowering operators $L_j(K h_\lambda) = K h_{\lambda-\epsilon_j}$

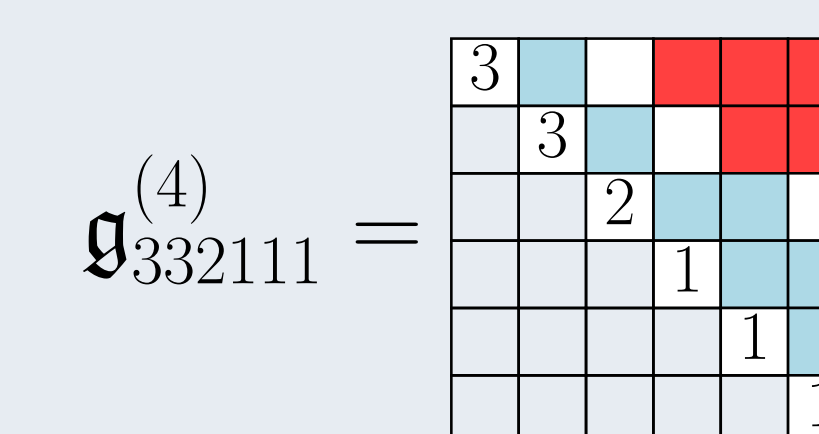


- non-roots of Ψ , roots of \mathcal{L}



$$\begin{aligned} K(\Psi; \mathcal{L}; 54332) &= (1 - L_4)^2 (1 - L_5)^2 \\ &\quad \cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) K h_{54332} \end{aligned}$$

$g_\lambda^{(k)} := K(\Psi; \mathcal{L}; \lambda)$ with $\text{band}(\Psi) = k$, $\text{band}(\mathcal{L}) = k+1$



Theorem: Shift Invariance

$$G_{1^\ell}^\perp g_{\lambda+1^\ell}^{(k+1)} = g_\lambda^{(k)}$$

Corollary

$g_\lambda^{(k)}$ branching follows from dual Pieri rule.

Conjecture

$$g_\lambda^{(k)} = g_\lambda^{(k)}$$