A Window into Symmetric Function Theory

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UVA Math Club
Lightning Round

2 March 2021

Symmetric Group

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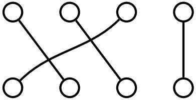

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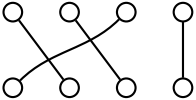
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} =$$ 

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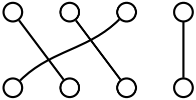

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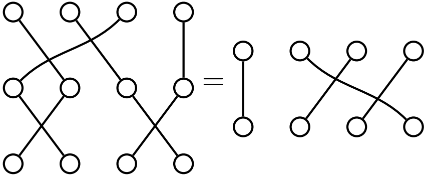

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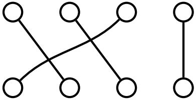

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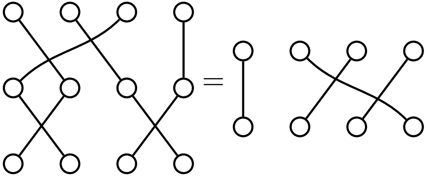

- S_n is a "group"

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- $f \in \mathbb{Q}[x_1, \ldots, x_n]$ multivariate polynomial

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- $\sigma \in S_n$ acts as $\sigma.f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$

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$$\begin{aligned} e_1 = x_1 + x_2 + x_3 = & h_1 \\ e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad & h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\ e_3 = x_1x_2x_3 \quad & h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \cdots \end{aligned}$$

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\end{aligned}
$$

- $\{f \in \mathbb{Q}[x_1, \dots, x_n] \mid \sigma.f = f \,\forall \sigma \in S_n\}$ forms a vector space, $\Lambda_{\mathbb{Q}}$.

Combinatorics of Symmetric Polynomials

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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Symmetric functions are polynomials in the $e_1, e_2, \ldots$, or in the $h_1, h_2, \ldots$

$$3h_2 h_1^2 - h_2^2 + 6h_3 h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

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Basis of $\Lambda_{\mathbb{Q}}$?

Partitions

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

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$$5 \to \ydiagram{5}$$

$$4+1 \to \ydiagram{4,1}$$

$$3+2 \to \ydiagram{3,2}$$

$$3+1+1 \to \ydiagram{3,1,1}$$

$$2+2+1 \to \ydiagram{2,2,1}$$

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1. How many partitions of n? No known closed-form formula!
2. Many interesting connections to number theory (Ramanujan).
3. Generating function for $p(n) =$ number of partitions of n is inverse of Euler ϕ function.

Tableaux

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Filling of partition diagram of λ with numbers such that

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For $\lambda = (2, 1)$,

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Schur functions

Associate a polynomial to SSYT(λ).

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$$s_{(21)}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + 2x_1x_2x_3$$

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For λ a partition

$$s_\lambda = \sum_{T \in \mathrm{SSYT}} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

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- Schur functions form a basis for $\Lambda_{\mathbb{Q}}$

Why Schur functions?

Harmonic polynomials

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$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

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$$\begin{aligned} M &= \mathrm{sp}\left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c\right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\qquad x_3 - x_1, x_2 - x_3, 1\} \end{aligned}$$

Harmonic polynomials

1. S_3 action on M fixes vector subspaces!

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Schur basis expansion counts multiplicity of irreducible S_n fixed subspaces!

Schur positivity

Upshot

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1. Schur functions $\leftrightarrow$ S_n-invariant subspaces.

Schur positivity

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1. Schur functions $\leftrightarrow$ S_n-invariant subspaces.
2. Via Frobenius characteristic map, questions about S_n-action on vector spaces get translated to questions about Schur expansion coefficients in symmetric functions.

Schur positivity

Interesting algebraic combinatorics questions

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1. Is a symmetric function Schur positive?

Schur positivity

Interesting algebraic combinatorics questions

1. Is a symmetric function Schur positive?
2. What do the Schur expansion coefficients count?

Getting more information

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Solution: minimal S_n-fixed subspace of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\ydiagram{1,1,1}} + q^2 s_{\ydiagram{2,1}} + q s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

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Minimal S_n-invariant subspace with bidegree $(a, b) \mapsto q^a t^b s_\lambda$

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$$\tilde{H}_\mu = qts_{\ydiagram{1,1,1}} + ts_{\ydiagram{2,1}} + qs_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

Diagonal harmonics

- Define ∇ by $\nabla \tilde{H}_\mu = B_\mu(q,t)\tilde{H}_\mu$ for eigenvalue $B_\mu(q,t) \in \mathbb{Q}[q,t]$.

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Open question

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What is the Schur expansion of ∇e_n?

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Recover earlier story by taking $t = 0$ and $y_i = 1$ for all y_i's.