Dens, nests, and Catalanimals: a walk through the zoo of shuffle theorems

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Polynomials

- $f \in \mathbb{Q}[x_1, \ldots, x_n]$ multivariate polynomial

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- $\sigma \in S_n$ acts as $\sigma.f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$

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- $\{f \in \mathbb{Q}[x_1, \ldots, x_n] \mid \sigma.f = f \;\forall \sigma \in S_n\}$ forms a vector space, $\Lambda_{\mathbb{Q}}$.

Combinatorics of Symmetric Polynomials

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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Symmetric functions are polynomials in the $e_1, e_2, \ldots$, or in the $h_1, h_2, \ldots$

$$3h_2 h_1^2 - h_2^2 + 6h_3 h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

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Basis of $\Lambda_{\mathbb{Q}}$?

Partitions

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

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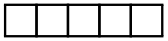
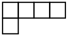
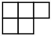
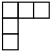
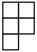
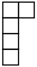
$5 \rightarrow$ 

$4 + 1 \rightarrow$ 

$3 + 2 \rightarrow$ 

$3 + 1 + 1 \rightarrow$ 

$2 + 2 + 1 \rightarrow$ 

$2 + 1 + 1 + 1 \rightarrow$ 

$1 + 1 + 1 + 1 + 1 \rightarrow$ 

Tableaux

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Collection is called SSYT(λ).

For $\lambda = (2, 1)$,

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Schur functions

Associate a polynomial to SSYT(λ).

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Weight:	(2,1,0)	(2,0,1)	(0,2,1)	(1,2,0)	(1,0,2)	(0,1,2)	(1,1,1)	(1,1,1)

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$$s_{(21)}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + 2x_1x_2x_3$$

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$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

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- s_λ is a symmetric function
- Schur functions form a basis for $\Lambda_{\mathbb{Q}}$

Why Schur functions?

Harmonic polynomials

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Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

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$$\begin{aligned} M &= \mathrm{sp}\left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\qquad x_3 - x_1, x_2 - x_3, 1\} \end{aligned}$$

Harmonic polynomials

1. S_3 action on M fixes vector subspaces!

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Schur basis expansion counts multiplicity of irreducible S_n fixed subspaces!

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Via Frobenius characteristic map, questions about S_n-representations get translated to questions about Schur expansion coefficients in symmetric functions.

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Via Frobenius characteristic map, questions about S_n-representations get translated to questions about Schur expansion coefficients in symmetric functions.

Does a symmetric function expand into Schur basis with nonnegative coefficients? Is there a combinatorial description for coefficients?

Getting more information

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Break M up into smallest S_n fixed subspaces

$$\mathrm{sp}\{\Delta\} \oplus \mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\} \oplus \mathrm{sp}\{x_3-x_1, x_2-x_3\} \oplus \mathrm{sp}\{1\}$$

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Solution: minimal S_n-fixed subspace of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\ydiagram{1,1,1}} + q^2 s_{\ydiagram{2,1}} + q s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

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$$?? = q^3 s_{(1,1,1)} + q^2 s_{(2,1)} + q s_{(2,1)} + s_{(3)}$$

Answer: "Hall-Littlewood polynomial" $H_{(1,1,1)}(X; q)$.

A Problem

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- Does there exist a family of S_n-representations whose (bigraded) Frobenius characteristics equal $\tilde{H}_\lambda(X;q,t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

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Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X; q, t)$

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- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$. (Still open!)

Garsia-Haiman modules

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \mathrm{sp}\{f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \mid \left(\sum_{j=1}^{n} \partial_{x_j}^r \partial_{y_j}^s\right) f = 0, \forall r + s > 0\}$$

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Question

What symmetric function is the bigraded Frobenius characteristic of DH_n?

∇e_n

Frobenius characteristic of DH_3

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$$= \frac{t^3\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2t+qt^2+qt)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{q^3\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

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$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q+t+1)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

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Operator ∇

$$\nabla \tilde{H}_\lambda(X;q,t) = q^{n(\lambda)} t^{n(\lambda')} \tilde{H}_\lambda(X;q,t)$$

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Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n.

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\mathsf{area}(\lambda)} q^{\mathsf{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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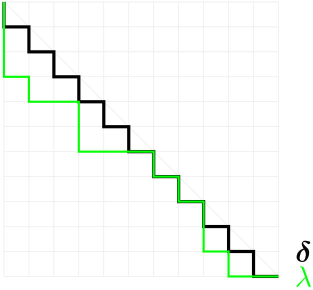
Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



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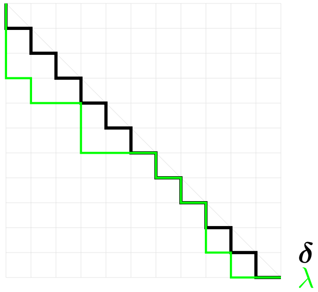

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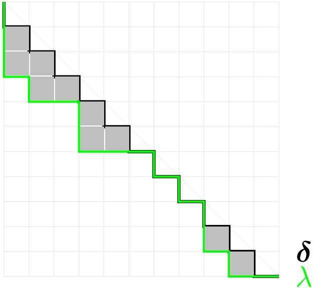

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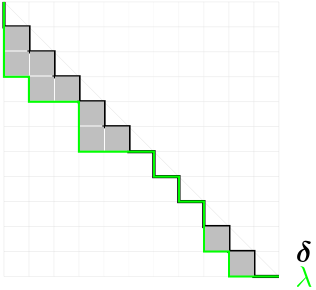

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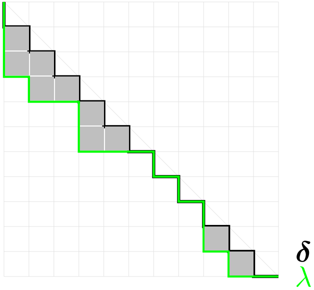

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dinv

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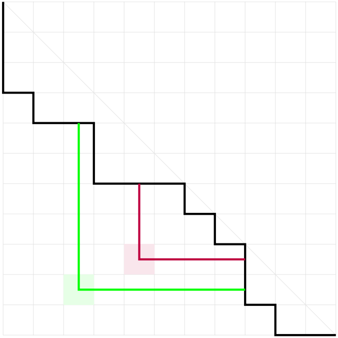

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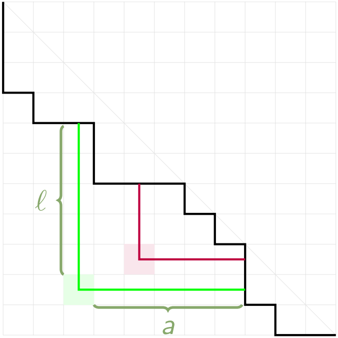

Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

LLT Polynomials

Defined in general for a tuple of skew shapes $\nu = (\nu^{(1)}, \ldots, \nu^{(r)})$

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- $\mathcal{G}_\nu$ is Schur-positive for any tuple of skew shapes ν [Grojnowski-Haiman, 2007].

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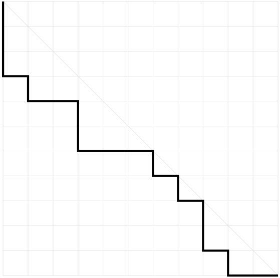

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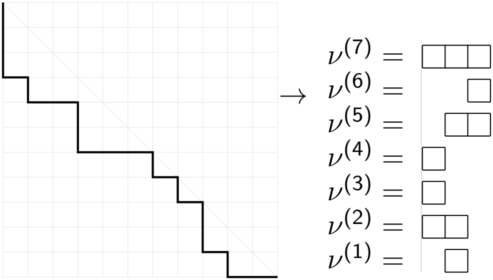

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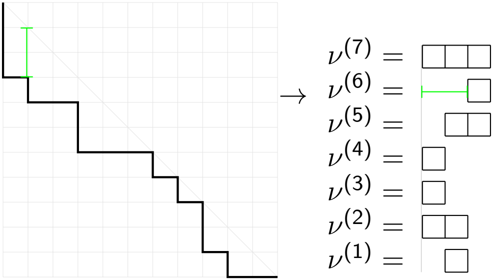

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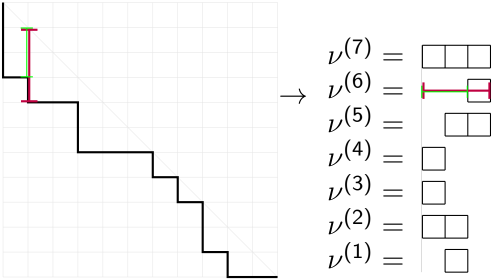

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- $$\mathcal{G}_{\square\square,\,\square}(x_1, x_2; q) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + q x_1^2 x_2 + q x_1 x_2^2$$

$$\begin{array}{cccccc} \boxed{1\,|\,1} & \boxed{1\,|\,2} & \boxed{1\,|\,2} & \boxed{2\,|\,2} & \boxed{1\,|\,1} & \boxed{2\,|\,2} \\ \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{1} \end{array}$$

$$= s_3 + q s_{2,1}$$

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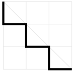
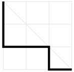
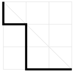
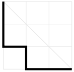
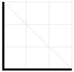
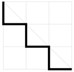
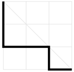
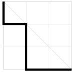
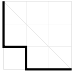
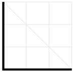
		
		
		
		
		

Example ∇e_3

λ	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)}$	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	
	$q^2 t$	
	qt	
	qt^2	
	t^3	

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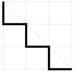
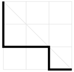
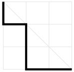
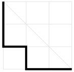
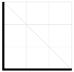
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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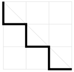
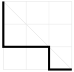
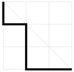
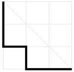
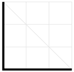
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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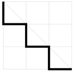
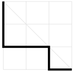
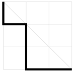
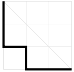
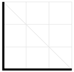
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

- Entire quantity is q, t-symmetric
- Coefficient of $s_{1,1,1}$ in sum is a "(q, t)-Catalan number" $(q^3 + q^2 t + qt + qt^2 + t^3)$.

Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

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Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2016)

For m, n coprime, the operator $e_k[-MX^{m,n}]$ acting on Λ satisfies

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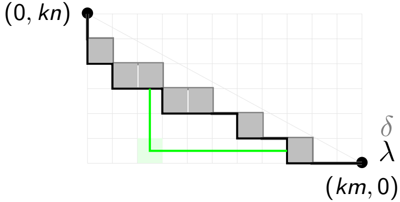

Welcome to the Zoo

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- $\mathcal{E}$ contains subalgebra $\Lambda(X^{m,n}) \cong \Lambda$ for each coprime pair $(m, n) \in \mathbb{Z}^2$.
- In general, $\mathcal{E}$-action can be a pain to compute in a nice way, but sometimes it is nice!

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- We can take "polynomial part" (restrict to only polynomial GL_l-characters) to get a symmetric function.

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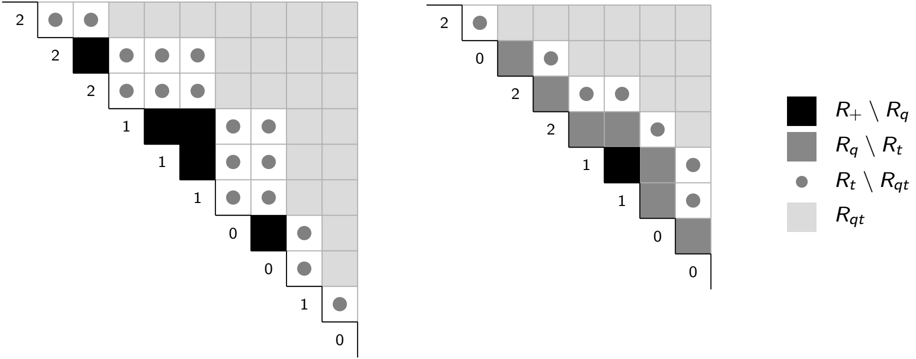

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- In this case, we set $\operatorname{cub}(H) = f$.
- The cuddly conditions allow a nice coproduct formula for $f[X + Y]$ in terms of cubs of "restrictions" of H.

Cuddly Catalanimals with cub e_k

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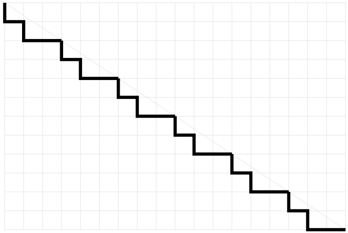

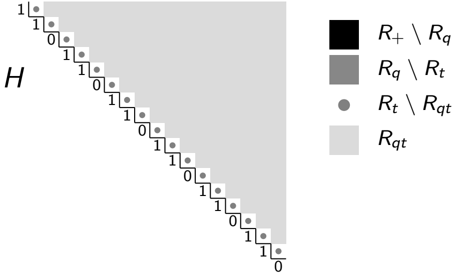

$\delta = (1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0)$ and
$e_6[-MX^{3,2}] \cdot 1 = \omega \, \mathrm{pol}_X \, H$

$1,1$-Cuddly Catalanimals with cub s_μ

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- $\mu = \begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \; & \; & \; \\ \cline{1-3} \; & \; & \; \\ \cline{1-3}\end{array} \to \begin{array}{|c|c|c|c|}\hline + & . & . & - \\ \hline + & . & - \\ \cline{1-3} + & . & - \\ \cline{1-3}\end{array} \to \begin{array}{|c|c|c|c|}\hline 1 & 0 & 1 & 0 \\ \hline 2 & 1 & 0 \\ \cline{1-3} 2 & 2 & 1 \\ \cline{1-3}\end{array}.$

1, 1-Cuddly Catalanimals with cub s_μ

- Can construct root sets and weight from the content diagonals of μ.
- $\mu =$

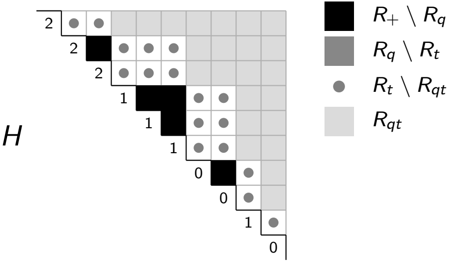
$\rightarrow$

+	.	.	−
+	.	−	
+	.	−	

$\rightarrow$

1	0	1	0
2	1	0	
2	2	1	

.



$s_\mu[-MX^{1,1}] \cdot 1 = \nabla s_\mu = \omega\, \mathrm{pol}_X\, H$ (up to q, t-monomial)

Dens and nests

Theorem (Blasiak-Haiman-Morse-Pun-S. (2021^+))

For every partition μ and coprime positive integers m, n, we have

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- Combinatorial RHS: Over all *nests* π in a *den* associated to μ and m, n.
- Conjectured by Loehr-Warrington (2008) when $n = 1$ with different combinatorics (but bijectively related).

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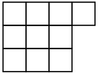
$\mu =$ 

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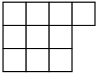
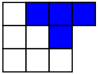
$\mu =$ 

- $p(\mu) =$ number of boxes with positive content. 

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For every partition μ and coprime positive integers m, n, we have

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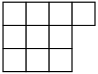
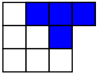
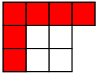
$\mu =$ 

- $p(\mu) =$ number of boxes with positive content. 
- $h = m(\text{largest hook length in } \mu) = m(\mu_1 + \ell(\mu) - 1)$. 

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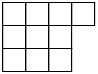
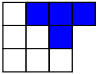
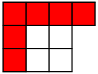
$\mu =$ 

- $p(\mu) =$ number of boxes with positive content. 
- $h = m(\text{largest hook length in } \mu) = m(\mu_1 + \ell(\mu) - 1)$. 

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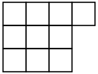
$\mu =$ 

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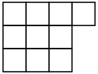
$\mu =$ 

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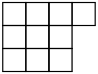
- $\mu =$  $\Longrightarrow \gamma = (1, 2, 3, 2, 1, 1)$.

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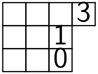
- $\mu =$  $\Longrightarrow \delta(\mu) = (1, 0, 1, 1, 0, 0, \ldots)$
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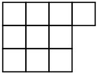
- $\mu =$  $\Longrightarrow \epsilon(\mu) = (0, 0, 0, 0, 1, 1, \ldots)$

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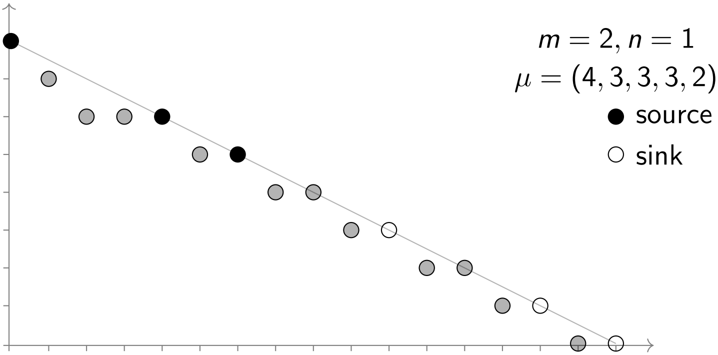

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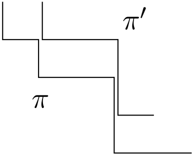
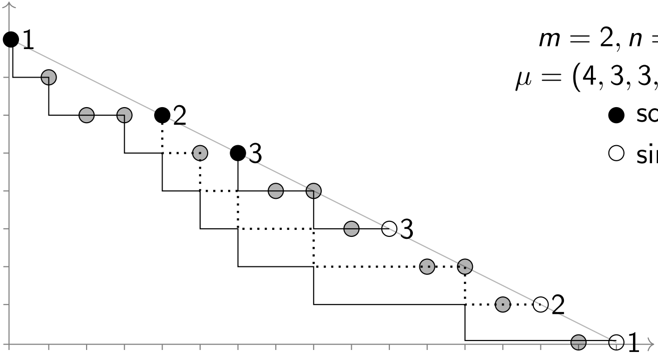


Dens and nests

Example of the "highest nest" π^0



$$m = 2, n = 1$$
$$\mu = (4, 3, 3, 3, 2)$$

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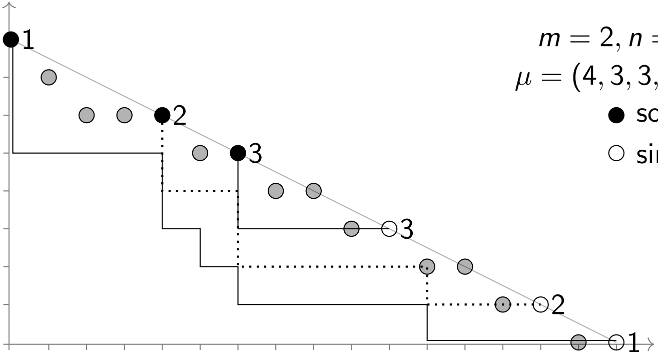
Dens and nests

Example of another nest.



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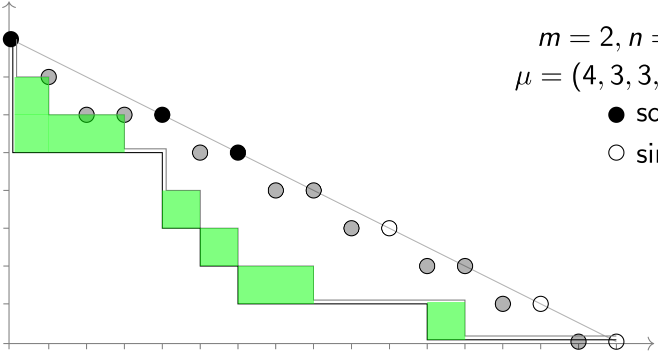

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$\text{area}(\pi_1) = 9$

Dens and nests

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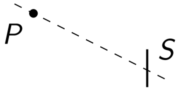

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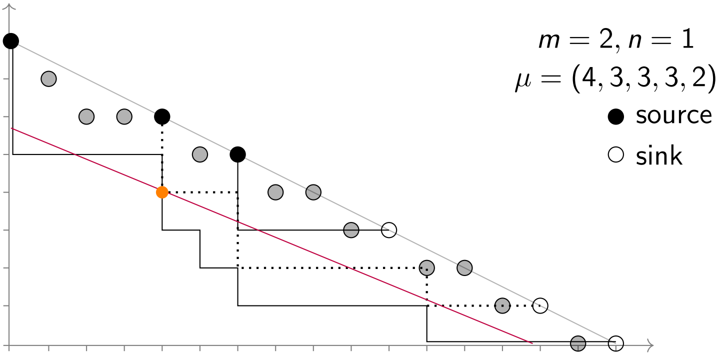

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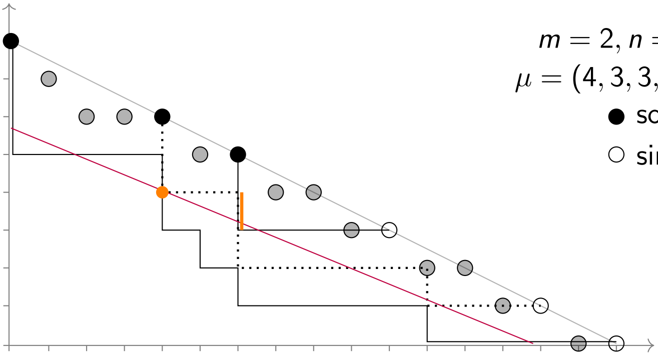
dinv



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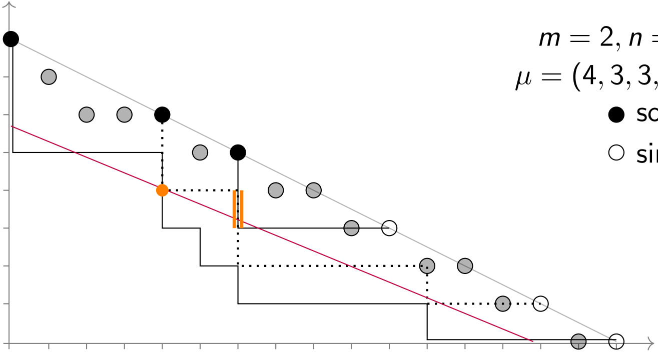

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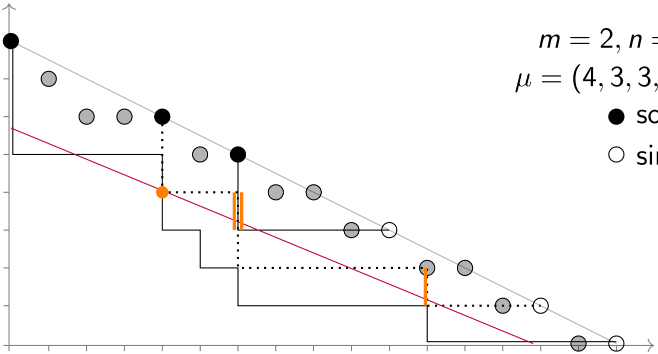

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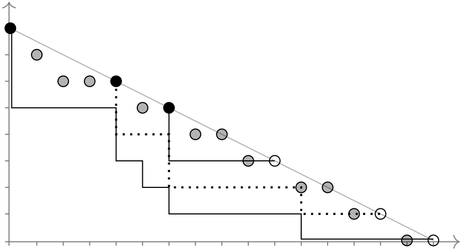

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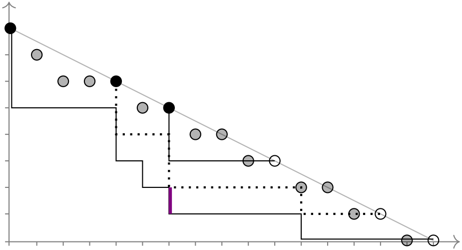

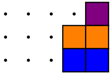

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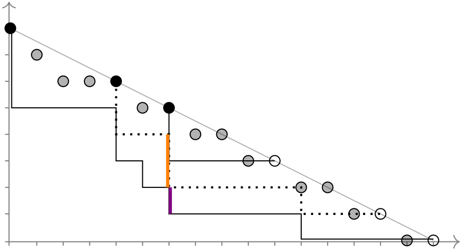

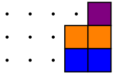

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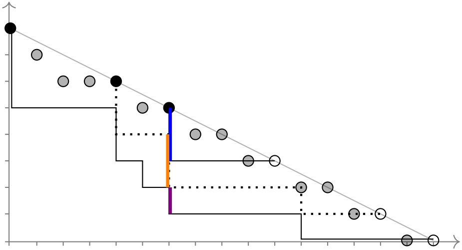

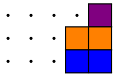

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- In our paper, we provide a more general definition of den as a tuple of data $(h, p, d, e) \in \mathbb{Z}_{>0} \times (\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{Z}^{h+1} \times \mathbb{Z}^{h+1}$ subject to some conditions.

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- This allows us to simultaneously generalize the $s_\lambda[-MX^{m,n}]$ formula and our "shuffle theorem for paths under any line" formula (BHMPS).

Other exhibits for next time

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- Special cases include Schur functions and Hall-Littlewood polynomials.
- Unicorn Catalanimals (or Catalan functions) where $R_t = R_{qt} = \varnothing$ also have a rich (older) results and combinatorics, but served as inspiration. (Chen-Haiman, Blasiak-Morse-Pun-Summers, Blasiak-Morse-Pun)

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Future work: exit through the gift shop

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- What other families of symmetric functions can be represented by Catalanimals? Upcoming: Macdonald polynomials
- What connections do Catalanimals have with machinery used to prove other shuffle theorems, such as work by Carlsson-Mellit?

Thank you for visiting!

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