A Raising Operator Formula for Macdonald Polynomials and other related families

George H. Seelinger

joint work with J. Blasiak, M. Haiman, J. Morse, and A. Pun

ghseeli@umich.edu

Michigan State University Combinatorics and Graph Theory Seminar

30 January 2025

1. **Background on symmetric functions and Macdonald polynomials**
2. Shuffle theorems, combinatorics, and LLT polynomials
3. A new formula for Macdonald polynomials

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q, t)[x_1, \ldots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q,t)[x_1, \ldots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q,t)[x_1, \ldots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

- E.g. for $n = 3$,

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

$$e_3 = x_1 x_2 x_3 \quad h_3 = x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + \cdots$$

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q,t)[x_1, \ldots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

- E.g. for $n = 3$,

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

$$e_3 = x_1 x_2 x_3 \quad h_3 = x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + \cdots$$

- Let $\Lambda = \mathbb{Q}(q,t)[e_1, e_2, \ldots] = \mathbb{Q}(q,t)[h_1, h_2, \ldots]$. Call these "symmetric functions."

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q,t)[x_1, \ldots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

- E.g. for $n = 3$,

$$e_1 = x_1 + x_2 + x_3 = h_1$$
$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$
$$e_3 = x_1 x_2 x_3 \quad h_3 = x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + \cdots$$

- Let $\Lambda = \mathbb{Q}(q,t)[e_1, e_2, \ldots] = \mathbb{Q}(q,t)[h_1, h_2, \ldots]$. Call these "symmetric functions."
- Λ is a $\mathbb{Q}(q,t)$-algebra.

Bases for symmetric functions

Dimension of degree d symmetric functions?

Bases for symmetric functions

Dimension of degree d symmetric functions? Number of partitions of d.

Bases for symmetric functions

Dimension of degree d symmetric functions? Number of partitions of d.

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

Bases for symmetric functions

Dimension of degree d symmetric functions? Number of partitions of d.

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

$5 \rightarrow$

$4 + 1 \rightarrow$

$3 + 2 \rightarrow$

$3 + 1 + 1 \rightarrow$

$2 + 2 + 1 \rightarrow$

$2 + 1 + 1 + 1 \rightarrow$

$1 + 1 + 1 + 1 + 1 \rightarrow$

Bases for symmetric functions

Dimension of degree d symmetric functions? Number of partitions of d.

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

$5 \rightarrow$ ▢▢▢▢▢

$4 + 1 \rightarrow$

$3 + 2 \rightarrow$

$3 + 1 + 1 \rightarrow$

$2 + 2 + 1 \rightarrow$

$2 + 1 + 1 + 1 \rightarrow$

$1 + 1 + 1 + 1 + 1 \rightarrow$

$\implies$ any basis of symmetric functions is indexed by partitions.

Young Tableaux

Definition

Filling of partition diagram of λ with numbers such that

Young Tableaux

Definition

Filling of partition diagram of λ with numbers such that

1. strictly increasing up columns

Definition

Filling of partition diagram of λ with numbers such that

1. strictly increasing up columns
2. weakly increasing along rows

Definition

Filling of partition diagram of λ with numbers such that

1. strictly increasing up columns
2. weakly increasing along rows

Collection is called $\text{SSYT}(\lambda)$.

Definition

Filling of partition diagram of λ with numbers such that

1. strictly increasing up columns
2. weakly increasing along rows

Collection is called $SSYT(\lambda)$.

For $\lambda = (2, 1)$,

$$\begin{array}{c} \boxed{2} \\ \boxed{1\,1} \end{array}, \quad \begin{array}{c} \boxed{3} \\ \boxed{1\,1} \end{array}, \quad \begin{array}{c} \boxed{3} \\ \boxed{2\,2} \end{array}, \quad \begin{array}{c} \boxed{2} \\ \boxed{1\,2} \end{array}, \quad \begin{array}{c} \boxed{3} \\ \boxed{1\,3} \end{array}, \quad \begin{array}{c} \boxed{3} \\ \boxed{2\,3} \end{array}, \quad \begin{array}{c} \boxed{2} \\ \boxed{1\,3} \end{array}, \quad \begin{array}{c} \boxed{3} \\ \boxed{1\,2} \end{array}$$

Polynomials from tableaux

Associate a polynomial to SSYT(λ).

Polynomials from tableaux

Associate a polynomial to SSYT(λ).

$$\begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 3 & \\ \hline 2 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 3 & \\ \hline 2 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}$$

Polynomials from tableaux

Associate a polynomial to SSYT(λ).

$$\begin{array}{|c|c|} \hline 2 \\ \hline 1 & 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 3 \\ \hline 1 & 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 3 \\ \hline 2 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 2 \\ \hline 1 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 3 \\ \hline 1 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 3 \\ \hline 2 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 2 \\ \hline 1 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 3 \\ \hline 1 & 2 \\ \hline \end{array}$$

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Polynomials from tableaux

Associate a polynomial to SSYT(λ).

$$\begin{array}{c} \boxed{2} \\ \boxed{1}\,\boxed{1} \end{array}, \quad \begin{array}{c} \boxed{3} \\ \boxed{1}\,\boxed{1} \end{array}, \quad \begin{array}{c} \boxed{3} \\ \boxed{2}\,\boxed{2} \end{array}, \quad \begin{array}{c} \boxed{2} \\ \boxed{1}\,\boxed{2} \end{array}, \quad \begin{array}{c} \boxed{3} \\ \boxed{1}\,\boxed{3} \end{array}, \quad \begin{array}{c} \boxed{3} \\ \boxed{2}\,\boxed{3} \end{array}, \quad \begin{array}{c} \boxed{2} \\ \boxed{1}\,\boxed{3} \end{array}, \quad \begin{array}{c} \boxed{3} \\ \boxed{1}\,\boxed{2} \end{array}$$

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Definition

For λ a partition

$$s_\lambda = \sum_{T \in \mathsf{SSYT}(\lambda)} \mathbf{x}^T \text{ for } \mathbf{x}^T = \prod_{i \in T} x_i$$

Polynomials from tableaux

Associate a polynomial to SSYT(λ).

$$\begin{array}{c} \boxed{2} \\ \boxed{1}\boxed{1} \end{array}, \begin{array}{c} \boxed{3} \\ \boxed{1}\boxed{1} \end{array}, \begin{array}{c} \boxed{3} \\ \boxed{2}\boxed{2} \end{array}, \begin{array}{c} \boxed{2} \\ \boxed{1}\boxed{2} \end{array}, \begin{array}{c} \boxed{3} \\ \boxed{1}\boxed{3} \end{array}, \begin{array}{c} \boxed{3} \\ \boxed{2}\boxed{3} \end{array}, \begin{array}{c} \boxed{2} \\ \boxed{1}\boxed{3} \end{array}, \begin{array}{c} \boxed{3} \\ \boxed{1}\boxed{2} \end{array}$$

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2 x_1 x_2 x_3$$

Definition

For λ a partition

$$s_\lambda = \sum_{T \in \mathsf{SSYT}(\lambda)} \mathbf{x}^T \text{ for } \mathbf{x}^T = \prod_{i \in T} x_i$$

- s_λ is a symmetric function.

Polynomials from tableaux

Associate a polynomial to SSYT(λ).

$$\begin{array}{cc} 2 & \\ 1 & 1 \end{array}, \quad \begin{array}{cc} 3 & \\ 1 & 1 \end{array}, \quad \begin{array}{cc} 3 & \\ 2 & 2 \end{array}, \quad \begin{array}{cc} 2 & \\ 1 & 2 \end{array}, \quad \begin{array}{cc} 3 & \\ 1 & 3 \end{array}, \quad \begin{array}{cc} 3 & \\ 2 & 3 \end{array}, \quad \begin{array}{cc} 2 & \\ 1 & 3 \end{array}, \quad \begin{array}{cc} 3 & \\ 1 & 2 \end{array}$$

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Definition

For λ a partition

$$s_\lambda = \sum_{T \in \mathsf{SSYT}(\lambda)} \mathbf{x}^T \text{ for } \mathbf{x}^T = \prod_{i \in T} x_i$$

- s_λ is a symmetric function.
- $\{s_\lambda\}_\lambda$ forms a basis for $\Lambda_{\mathbb{Q}}$.

Frobenius charactersitc, Frob: $Rep(S_n) \to \Lambda$.

Representation theory and Schur functions

Frobenius charactersitc, Frob: $Rep(S_n) \to \Lambda$.

- Irreducible representations of S_n are labeled by partitions of n.

Representation theory and Schur functions

Frobenius charactersitc, Frob: $Rep(S_n) \to \Lambda$.

- Irreducible representations of S_n are labeled by partitions of n.
- Irreducible S_n-representation V_λ has $\mathrm{Frob}(V_\lambda) = s_\lambda$

Representation theory and Schur functions

Frobenius charactersitc, Frob: $Rep(S_n) \to \Lambda$.

- Irreducible representations of S_n are labeled by partitions of n.
- Irreducible S_n-representation V_λ has $\text{Frob}(V_\lambda) = s_\lambda$
- $U \cong V \oplus W \implies \text{Frob}(U) = \text{Frob}(V) + \text{Frob}(W)$

Representation theory and Schur functions

Frobenius charactersitc, Frob: $Rep(S_n) \to \Lambda$.

- Irreducible representations of S_n are labeled by partitions of n.
- Irreducible S_n-representation V_λ has $\mathrm{Frob}(V_\lambda) = s_\lambda$
- $U \cong V \oplus W \implies \mathrm{Frob}(U) = \mathrm{Frob}(V) + \mathrm{Frob}(W)$
- $\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}}(V \times W) \mapsto \mathrm{Frob}(V) \cdot \mathrm{Frob}(W)$

Representation theory and Schur functions

Frobenius charactersitc, Frob: $Rep(S_n) \to \Lambda$.

- Irreducible representations of S_n are labeled by partitions of n.
- Irreducible S_n-representation V_λ has $\text{Frob}(V_\lambda) = s_\lambda$
- $U \cong V \oplus W \implies \text{Frob}(U) = \text{Frob}(V) + \text{Frob}(W)$
- $\text{Ind}_{S_m \times S_n}^{S_{m+n}} (V \times W) \mapsto \text{Frob}(V) \cdot \text{Frob}(W)$
- Upshot: S_n-representations go to symmetric functions in structure preserving way.

Representation theory and Schur functions

Frobenius charactersitc, Frob: $Rep(S_n) \to \Lambda$.

- Irreducible representations of S_n are labeled by partitions of n.
- Irreducible S_n-representation V_λ has $\text{Frob}(V_\lambda) = s_\lambda$
- $U \cong V \oplus W \implies \text{Frob}(U) = \text{Frob}(V) + \text{Frob}(W)$
- $\text{Ind}_{S_m \times S_n}^{S_{m+n}}(V \times W) \mapsto \text{Frob}(V) \cdot \text{Frob}(W)$
- Upshot: S_n-representations go to symmetric functions in structure preserving way.

Hidden Guide: Schur Positivity

"Naturally occurring" symmetric functions which are non-negative (coefficients in $\mathbb{N}$) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

Harmonic polynomials

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Harmonic polynomials

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

Harmonic polynomials

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

M is the vector space given by

Harmonic polynomials

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

M is the vector space given by

$$
\begin{aligned}
M &= \mathrm{sp}\left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\
&= \mathrm{sp}\{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\
&\qquad x_3 - x_1, x_2 - x_3, 1 \}
\end{aligned}
$$

Harmonic polynomials

$\mathsf{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$

$$\mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

1. Break M up into irreducible S_n-representations.

Harmonic polynomials

$$\mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

1. Break M up into irreducible S_n-representations.

$$\underbrace{\mathrm{sp}\{\Delta\}}_{} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{} \oplus \underbrace{\mathrm{sp}\{1\}}_{}$$

Harmonic polynomials

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

1. Break M up into irreducible S_n-representations.

$$\underbrace{\text{sp}\{\Delta\}}_{\square} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\square} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\square} \oplus \underbrace{\text{sp}\{1\}}_{\square\square\square}$$

2. How many times does an irreducible S_n-representation occur?

Harmonic polynomials

$$\mathsf{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

① Break M up into irreducible S_n-representations.

$$\underbrace{\mathsf{sp}\{\Delta\}}_{} \oplus \underbrace{\mathsf{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{} \oplus \underbrace{\mathsf{sp}\{x_3-x_1, x_2-x_3\}}_{} \oplus \underbrace{\mathsf{sp}\{1\}}_{}$$

② How many times does an irreducible S_n-representation occur?
Frobenius:

Harmonic polynomials

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

1. Break M up into irreducible S_n-representations.

$$\underbrace{\text{sp}\{\Delta\}}_{\Box\Box\Box} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\Box\Box} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\Box\Box} \oplus \underbrace{\text{sp}\{1\}}_{\Box\Box\Box}$$

2. How many times does an irreducible S_n-representation occur? Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\Box} + s_{\Box\Box} + s_{\Box\Box} + s_{\Box\Box\Box}$$

Harmonic polynomials

$$\mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

1. Break M up into irreducible S_n-representations.

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\yng} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\yng} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\yng} \oplus \underbrace{\mathrm{sp}\{1\}}_{\yng}$$

2. How many times does an irreducible S_n-representation occur?
   Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\yng} + s_{\yng} + s_{\yng} + s_{\yng}$$

Remark: $M \cong \mathbb{C}[x_1, x_2, x_3] / (\mathbb{C}[x_1, x_2, x_3]_+^{S_3})$ is a "regular representation."

Getting more information

Break M up into smallest S_n fixed subspaces

$$\underbrace{\mathrm{sp}\{\Delta\}}_{} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\mathrm{deg}=2} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\mathrm{deg}=1} \oplus \underbrace{\mathrm{sp}\{1\}}_{}$$

Break M up into smallest S_n fixed subspaces

$$\underbrace{\mathrm{sp}\{\Delta\}}_{} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\deg=2} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\deg=1} \oplus \underbrace{\mathrm{sp}\{1\}}_{}$$

Solution: irreducible S_n-representation of polynomials of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\tiny\yng(1,1,1)} + q^2 s_{\tiny\yng(2,1)} + q s_{\tiny\yng(2,1)} + s_{\tiny\yng(3)}$$

Break M up into smallest S_n fixed subspaces

$$\underbrace{\mathrm{sp}\{\Delta\}}_{} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\deg=2} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\deg=1} \oplus \underbrace{\mathrm{sp}\{1\}}_{}$$

Solution: irreducible S_n-representation of polynomials of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\square} + q^2 s_{\square\square} + q s_{\square\square} + s_{\square\square\square}$$

Answer: Hall-Littlewood polynomial $H_{\square}(X; q)$.

A Problem

- In 1988, Macdonald introduces one basis of symmetric polynomials to rule them all!

A Problem

- In 1988, Macdonald introduces one basis of symmetric polynomials to rule them all!
- Coefficients in $\mathbb{Q}(q,t)$, specializations give Hall-Littlewood polynomials, Schur polynomials, and many other famous bases.

A Problem

- In 1988, Macdonald introduces one basis of symmetric polynomials to rule them all!
- Coefficients in $\mathbb{Q}(q, t)$, specializations give Hall-Littlewood polynomials, Schur polynomials, and many other famous bases.
- Defined by orthogonality and triangularity under a certain inner-product.

A Problem

- In 1988, Macdonald introduces one basis of symmetric polynomials to rule them all!
- Coefficients in $\mathbb{Q}(q, t)$, specializations give Hall-Littlewood polynomials, Schur polynomials, and many other famous bases.
- Defined by orthogonality and triangularity under a certain inner-product.
- Garsia modifies these polynomials so

  $$\tilde{H}_\lambda(X; q, t) = \sum_\mu \tilde{K}(q, t) s_\mu \text{ conjecturally satisfies } \tilde{K}(q, t) \in \mathbb{N}[q, t]$$

A Problem

- In 1988, Macdonald introduces one basis of symmetric polynomials to rule them all!
- Coefficients in $\mathbb{Q}(q, t)$, specializations give Hall-Littlewood polynomials, Schur polynomials, and many other famous bases.
- Defined by orthogonality and triangularity under a certain inner-product.
- Garsia modifies these polynomials so

  $$\tilde{H}_\lambda(X; q, t) = \sum_\mu \tilde{K}(q, t)s_\mu \text{ conjecturally satisfies } \tilde{K}(q, t) \in \mathbb{N}[q, t]$$

- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$.

A Problem

- In 1988, Macdonald introduces one basis of symmetric polynomials to rule them all!
- Coefficients in $\mathbb{Q}(q,t)$, specializations give Hall-Littlewood polynomials, Schur polynomials, and many other famous bases.
- Defined by orthogonality and triangularity under a certain inner-product.
- Garsia modifies these polynomials so

$$\tilde{H}_\lambda(X;q,t) = \sum_\mu \tilde{K}(q,t)s_\mu \text{ conjecturally satisfies } \tilde{K}(q,t) \in \mathbb{N}[q,t]$$

- $\tilde{H}_\lambda(X;1,1) = e_1^{|\lambda|}$.
- Does there exist a family of S_n-regular representations whose bigraded Frobenius characteristics equal $\tilde{H}_\lambda(X;q,t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): $M_\mu =$ span of partial derivatives of
  $\Delta_\mu = \det_{(i,j) \in \mu, k \in [n]}(x_k^{i-1} y_k^{j-1})$

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): $M_\mu =$ span of partial derivatives of
  $\Delta_\mu = \det_{(i,j) \in \mu, k \in [n]}(x_k^{i-1} y_k^{j-1})$

$$\Delta_{\scriptsize\begin{array}{c}\blacksquare\\\square\square\end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): $M_\mu =$ span of partial derivatives of
  $$\Delta_\mu = \det_{(i,j) \in \mu, k \in [n]}(x_k^{i-1} y_k^{j-1})$$

$$\Delta_{\square} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\mathrm{sp}\{\Delta_{2,1}\}}_{\deg=(1,1)} \oplus \underbrace{\mathrm{sp}\{y_3 - y_1, y_1 - y_2\}}_{\deg=(0,1)} \oplus \underbrace{\mathrm{sp}\{x_3 - x_1, x_1 - x_2\}}_{\deg=(1,0)} \oplus \underbrace{\mathrm{sp}\{1\}}_{\deg=(0,0)}$$

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): M_μ = span of partial derivatives of
  $\Delta_\mu = \det_{(i,j)\in\mu, k\in[n]}(x_k^{i-1} y_k^{j-1})$

$$\Delta_{\square} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\mathrm{sp}\{\Delta_{2,1}\}}_{\deg=(1,1)} \oplus \underbrace{\mathrm{sp}\{y_3 - y_1, y_1 - y_2\}}_{\deg=(0,1)} \oplus \underbrace{\mathrm{sp}\{x_3 - x_1, x_1 - x_2\}}_{\deg=(1,0)} \oplus \underbrace{\mathrm{sp}\{1\}}_{\deg=(0,0)}$$

Irreducible S_n-representation with bidegree $(a, b) \mapsto q^a t^b s_\lambda$

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): M_μ = span of partial derivatives of
  $\Delta_\mu = \det_{(i,j)\in\mu, k\in[n]}(x_k^{i-1} y_k^{j-1})$

$$\Delta_{\square} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\mathsf{sp}\{\Delta_{2,1}\}}_{\deg=(1,1)} \oplus \underbrace{\mathsf{sp}\{y_3 - y_1, y_1 - y_2\}}_{\deg=(0,1)} \oplus \underbrace{\mathsf{sp}\{x_3 - x_1, x_1 - x_2\}}_{\deg=(1,0)} \oplus \underbrace{\mathsf{sp}\{1\}}_{\deg=(0,0)}$$

Irreducible S_n-representation with bidegree $(a, b) \mapsto q^a t^b s_\lambda$

$$\tilde{H}_{\square} = qts_{\square} + ts_{\square} + qs_{\square} + s_{\square\square\square}$$

Garsia-Haiman modules

Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X; q, t)$

Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X; q, t)$

- Proved via connection to the Hilbert Scheme $Hilb^n(\mathbb{C}^2)$.

Garsia-Haiman modules

Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X; q, t)$

- Proved via connection to the Hilbert Scheme $Hilb^n(\mathbb{C}^2)$.

Corollary

$\tilde{H}_\lambda(X; q, t) = \sum_\mu \tilde{K}_{\lambda\mu}(q, t) s_\mu$ *satisfies* $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$.

Garsia-Haiman modules

Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X; q, t)$

- Proved via connection to the Hilbert Scheme $Hilb^n(\mathbb{C}^2)$.

Corollary

$\tilde{H}_\lambda(X; q, t) = \sum_\mu \tilde{K}_{\lambda\mu}(q, t)s_\mu$ *satisfies* $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$.

- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	$SSYT(\lambda)$
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	??

Garsia-Haiman modules

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \mathrm{sp}\{f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \mid \left(\sum_{j=1}^{n} \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

Garsia-Haiman modules

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \mathrm{sp}\{f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \mid \left(\sum_{j=1}^{n} \partial_{x_j}^r \partial_{y_j}^s\right) f = 0, \forall r + s > 0\}$$

Question

What symmetric function is the bigraded Frobenius characteristic of DH_n?

Frobenius characteristic of DH_3

Frobenius characteristic of DH_3

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2t+qt^2+qt)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{q^3 \tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

Frobenius characteristic of DH_3

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2t+qt^2+qt)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{q^3\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q+t+1)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

Frobenius characteristic of DH_3

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2t+qt^2+qt)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{q^3\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q+t+1)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

Operator ∇

$$\nabla\tilde{H}_\lambda(X; q, t) = q^{n(\lambda)}t^{n(\lambda^*)}\tilde{H}_\lambda(X; q, t),$$

where $n(\lambda) = \sum_i(i - 1)\lambda_i$ and λ^* is the transpose partition to λ.

∇e_n

Frobenius characteristic of DH_3

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2t+qt^2+qt)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{q^3 \tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q+t+1)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

Operator ∇

$$\nabla \tilde{H}_\lambda(X; q, t) = q^{n(\lambda)} t^{n(\lambda^*)} \tilde{H}_\lambda(X; q, t),$$

where $n(\lambda) = \sum_i (i-1)\lambda_i$ and λ^* is the transpose partition to λ.

Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n.

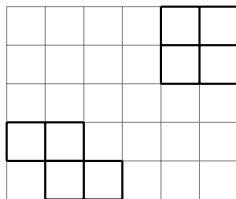# Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	$\text{SSYT}(\lambda)$
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	??
∇e_n	DH_n	Shuffle theorem

1. Background on symmetric functions and Macdonald polynomials
2. **Shuffle theorems, combinatorics, and LLT polynomials**
3. A new formula for Macdonald polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)

Key Object: LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)

- The *content* of a box in row y, column x is $x - y$.

$$\boldsymbol{\nu} = \left(, \right)$$

-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

Key Object: LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.

$$\boldsymbol{\nu} = \left(\quad , \quad \right)$$

Key Object: LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)

- The *content* of a box in row y, column x is $x - y$.

- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.

- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
  - $\mathrm{content}(b) = \mathrm{content}(a)$, or
  - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

Key Object: LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)

- The *content* of a box in row y, column x is $x - y$.

- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.

- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
  - $\mathrm{content}(b) = \mathrm{content}(a)$, or
  - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

$$\boldsymbol{\nu} = \left(\; \boxed{} \; , \; \boxed{} \; \right)$$

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

Key Object: LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
    - $\mathrm{content}(b) = \mathrm{content}(a)$, or
    - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

$$\boldsymbol{\nu} = \left(\; \rule{0pt}{0pt} \;, \; \rule{0pt}{0pt} \; \right)$$

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

Key Object: LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)

- The *content* of a box in row y, column x is $x - y$.

- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.

- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
    - $\text{content}(b) = \text{content}(a)$, or
    - $\text{content}(b) = \text{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

$$\boldsymbol{\nu} = \left(\rule{0pt}{0pt}\hspace{1.5cm}, \hspace{1.5cm} \right)$$

Attacking pairs: $(b_2, b_3), (b_3, b_4), \textcolor{red}{(b_4, b_5)}, (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

Key Object: LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
    - $\mathrm{content}(b) = \mathrm{content}(a)$, or
    - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), \textcolor{red}{(b_4, b_6)}, (b_5, b_7), (b_6, b_7), (b_7, b_8)$

Key Object: LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
  - $\mathrm{content}(b) = \mathrm{content}(a)$, or
  - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

Key Object: LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)

- The *content* of a box in row y, column x is $x - y$.

- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.

- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
  - $\mathrm{content}(b) = \mathrm{content}(a)$, or
  - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

Key Object: LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)

- The *content* of a box in row y, column x is $x - y$.

- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.

- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
  - $\mathrm{content}(b) = \mathrm{content}(a)$, or
  - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

$$\boldsymbol{\nu} = \left(\text{⊞}, \text{⊞} \right)$$

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

LLT Polynomials

- A *semistandard tableau* on ν is a map $T \colon \nu \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.

- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes ν is

$$\mathcal{G}_{\nu}(\boldsymbol{x}; q) = \sum_{T \in \mathrm{SSYT}(\nu)} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \nu} x_{T(a)}$.

LLT Polynomials

- A *semistandard tableau* on ν is a map $T\colon \nu \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.

- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes ν is

$$\mathcal{G}_\nu(\boldsymbol{x}; q) = \sum_{T \in \mathsf{SSYT}(\nu)} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \nu} x_{T(a)}$.

LLT Polynomials

- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.

- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{x}; q) = \sum_{T \in \text{SSYT}(\boldsymbol{\nu})} q^{\text{inv}(T)} \boldsymbol{x}^T,$$

where $\text{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

$$T \quad = \quad$$

inversion

LLT Polynomials

- A *semistandard tableau* on ν is a map $T \colon \nu \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.

- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes ν is

$$\mathcal{G}_{\nu}(\boldsymbol{x}; q) = \sum_{T \in \mathsf{SSYT}(\nu)} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \nu} x_{T(a)}$.

inversion

LLT Polynomials

- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.

- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{x}; q) = \sum_{T \in \mathsf{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

$T =$ (tableau)   non-inversion

LLT Polynomials

- A *semistandard tableau* on ν is a map $T\colon \nu \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.

- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes ν is

$$\mathcal{G}_{\nu}(\boldsymbol{x}; q) = \sum_{T \in \mathrm{SSYT}(\nu)} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \nu} x_{T(a)}$.

$$T \quad = \quad$$

non-inversion

LLT Polynomials

- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_{+}$ which restricts to a semistandard tableau on each $\nu_{(i)}$.

- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{x}; q) = \sum_{T \in \mathsf{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{x}^{T},$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^{T} = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

LLT Polynomials

- A *semistandard tableau* on ν is a map $T \colon \nu \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.

- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes ν is

$$\mathcal{G}_\nu(\boldsymbol{x}; q) = \sum_{T \in \mathsf{SSYT}(\nu)} q^{\mathsf{inv}(T)} \boldsymbol{x}^T,$$

where $\mathsf{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \nu} x_{T(a)}$.

inversion

$$\mathsf{inv}(T) = 4, \quad \boldsymbol{x}^T = x_1^2 x_2 x_3 x_4 x_5^2 x_6$$

- $\mathcal{G}_\nu(X; q)$ is a symmetric function

- $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$ is a symmetric function
- $\mathcal{G}_{\boldsymbol{\nu}}(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$

LLT Polynomials $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$

- $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$ is a symmetric function
- $\mathcal{G}_{\boldsymbol{\nu}}(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$
- $\mathcal{G}_{\boldsymbol{\nu}}$ were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of $U_q(\hat{\mathfrak{sl}}_r)$

LLT Polynomials $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$

- $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$ is a symmetric function
- $\mathcal{G}_{\boldsymbol{\nu}}(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$
- $\mathcal{G}_{\boldsymbol{\nu}}$ were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of $U_q(\hat{\mathfrak{sl}}_r)$
- When $\nu^{(i)}$ are partitions, the Schur-expansion coefficients are essentially parabolic Kazdhan-Luzstig polynomials.

- $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$ is a symmetric function
- $\mathcal{G}_{\boldsymbol{\nu}}(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$
- $\mathcal{G}_{\boldsymbol{\nu}}$ were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of $U_q(\hat{\mathfrak{sl}}_r)$
- When $\nu^{(i)}$ are partitions, the Schur-expansion coefficients are essentially parabolic Kazdhan-Luzstig polynomials.
- $\mathcal{G}_{\boldsymbol{\nu}}$ is Schur-positive for any tuple of skew shapes $\boldsymbol{\nu}$ [Grojnowski-Haiman, 2007].

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Summation over all k-by-k Dyck paths.

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_\lambda t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Summation over all k-by-k Dyck paths.
- area(λ) and dinv(λ) statistics of Dyck paths.

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_\lambda t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Summation over all k-by-k Dyck paths.
- area(λ) and dinv(λ) statistics of Dyck paths.
- $\mathcal{G}_{\nu(\lambda)}(X; q)$ a symmetric LLT polynomial indexed by a tuple of offset (skew) rows.

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Summation over all k-by-k Dyck paths.
- area(λ) and dinv(λ) statistics of Dyck paths.
- $\mathcal{G}_{\nu(\lambda)}(X; q)$ a symmetric LLT polynomial indexed by a tuple of offset (skew) rows.
- ω an automorphism of symmetric functions: $\omega(s_\lambda) = s_{\lambda^*}$ for $\lambda^* = $ transpose of λ.

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Summation over all k-by-k Dyck paths.
- area(λ) and dinv(λ) statistics of Dyck paths.
- $\mathcal{G}_{\nu(\lambda)}(X; q)$ a symmetric LLT polynomial indexed by a tuple of offset (skew) rows.
- ω an automorphism of symmetric functions: $\omega(s_\lambda) = s_{\lambda^*}$ for $\lambda^* =$ transpose of λ.
- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.

δ
λ
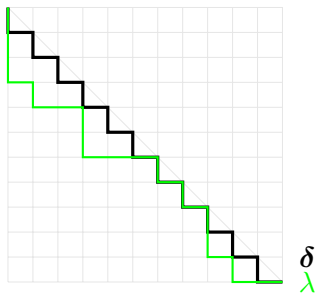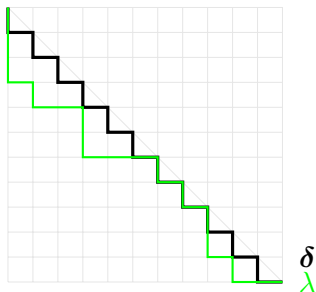
- area(λ) = number of squares above λ but below the path δ of alternating S-E steps.

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.
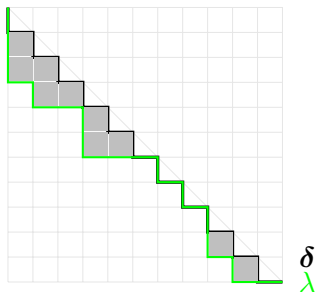
- area(λ) = number of squares above λ but below the path δ of alternating S-E steps.
- E.g., above area$(\lambda) = 10$.

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.
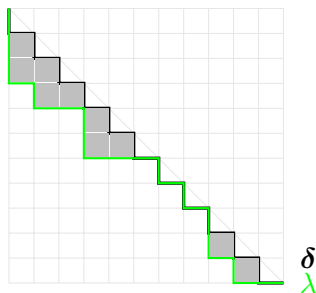
- area(λ) = number of squares above λ but below the path δ of alternating S-E steps.
- E.g., above area(λ) = 10.
- Catalan-number many Dyck paths for fixed k.

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.
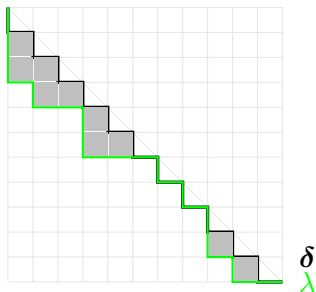
- area(λ) = number of squares above λ but below the path δ of alternating S-E steps.
- E.g., above area(λ) = 10.
- Catalan-number many Dyck paths for fixed k. (1,2,5,14,42,...)

dinv(λ) =# of balanced hooks in diagram below λ.
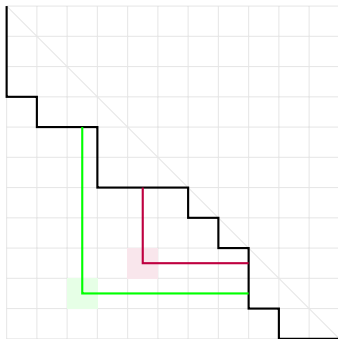
dinv(λ) =# of balanced hooks in diagram below λ.
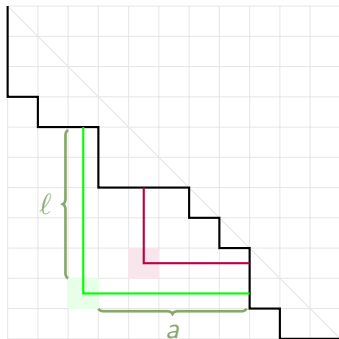
Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

Example ∇e_3

$$\lambda \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

Example ∇e_3

λ	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$

Example ∇e_3

λ	$q^{\mathsf{dinv}(\lambda)}t^{\mathsf{area}(\lambda)}$	$q^{\mathsf{dinv}(\lambda)}t^{\mathsf{area}(\lambda)}\omega\mathcal{G}_{\nu(\lambda)}(X;q^{-1})$
	q^3	
	$q^2 t$	
	qt	
	qt^2	
	t^3	

Example ∇e_3

| λ	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	$s_3 + q s_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$q t s_{2,1} + q^2 t s_{1,1,1}$
	qt	$t s_{2,1} + q t s_{1,1,1}$
	$q t^2$	$t^2 s_{2,1} + q t^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

Example ∇e_3

| λ	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	$s_3 + q s_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qt s_{2,1} + q^2 t s_{1,1,1}$
	qt	$t s_{2,1} + qt s_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

- Entire quantity is q, t-symmetric

Example ∇e_3

| λ	$q^{\mathsf{dinv}(\lambda)} t^{\mathsf{area}(\lambda)}$	$q^{\mathsf{dinv}(\lambda)} t^{\mathsf{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	$s_3 + q s_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$q t s_{2,1} + q^2 t s_{1,1,1}$
	$q t$	$t s_{2,1} + q t s_{1,1,1}$
	$q t^2$	$t^2 s_{2,1} + q t^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

- Entire quantity is q, t-symmetric
- Coefficient of $s_{1,1,1}$ in sum is a "(q, t)-Catalan number"
  $(q^3 + q^2 t + q t + q t^2 + t^3)$.

Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

When a problem is too difficult, try generalizing!

Algebraic Expression   Combinatorial Expression

$$\nabla e_k(X) \ = \sum q, t\text{-weighted Dyck paths}$$

Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

Algebraic Expression   Combinatorial Expression

$$\nabla e_k(X) \; = \sum q, t\text{-weighted Dyck paths}$$

Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2021)

For $m, n > 0$ coprime, the operator $e_k^{(m,n)}$ acting on Λ satisfies

$$e_k^{(m,n)} \cdot 1 = \sum q, t\text{-weighted } (km, kn)\text{-Dyck paths}$$

Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

Algebraic Expression   Combinatorial Expression

$$\nabla e_k(X) \; = \sum q, t\text{-weighted Dyck paths}$$

Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2021)

For $m, n > 0$ coprime, the operator $e_k^{(m,n)}$ acting on Λ satisfies

$$e_k^{(m,n)} \cdot 1 = \sum q, t\text{-weighted } (km, kn)\text{-Dyck paths}$$

$(0, kn)$

δ
λ

$(km, 0)$

Algebra $\mathcal{E} \curvearrowright \Lambda =$ symmetric polynomials

Algebra $\mathcal{E} \curvearrowright \Lambda =$ symmetric polynomials

$\mathcal{E}$ comes from algebraic geometry

Algebra $\mathcal{E} \curvearrowright \Lambda =$ symmetric polynomials

$\mathcal{E}$ comes from algebraic geometry

$$\mathcal{E} \cong \substack{\text{central} \\ \text{subalgebra}} \oplus \bigoplus_{m,n \text{ coprime}} \Lambda^{(m,n)}$$

Elliptic Hall Algebra

Algebra $\mathcal{E} \curvearrowright \Lambda =$ symmetric polynomials

$\mathcal{E}$ comes from algebraic geometry

$$\mathcal{E} \cong \begin{matrix} \text{central} \\ \text{subalgebra} \end{matrix} \oplus \bigoplus_{m,n \text{ coprime}} \Lambda^{(m,n)}$$

Elliptic Hall Algebra

Algebra $\mathcal{E} \curvearrowright \Lambda =$ symmetric polynomials

$\mathcal{E}$ comes from algebraic geometry

$$\mathcal{E} \cong \begin{array}{c}\text{central}\\\text{subalgebra}\end{array} \oplus \bigoplus_{m,n \text{ coprime}} \Lambda^{(m,n)}$$

$\Lambda^{(m,n)} \cong \begin{array}{c}\text{symmetric}\\\text{polynomials}\end{array}$

Elliptic Hall Algebra

Algebra $\mathcal{E} \curvearrowright \Lambda =$ symmetric polynomials

$\mathcal{E}$ comes from algebraic geometry

$\mathcal{E} \cong \begin{matrix} \text{central} \\ \text{subalgebra} \end{matrix} \oplus \bigoplus_{m,n \text{ coprime}} \Lambda^{(m,n)}$

$\Lambda^{(m,n)} \cong \begin{matrix} \text{symmetric} \\ \text{polynomials} \end{matrix}$

LHS of Shuffle Theorem $= e_k^{(1,1)} \in \Lambda^{(1,1)}$ acting on $1 \in \Lambda$.

LHS of Rational Shuffle Theorem $= e_k^{(m,n)} \in \Lambda^{(m,n)}$ acting on $1 \in \Lambda$.

Elliptic Hall Algebra

Algebra $\mathcal{E} \curvearrowright \Lambda =$ symmetric polynomials

$\mathcal{E}$ comes from algebraic geometry

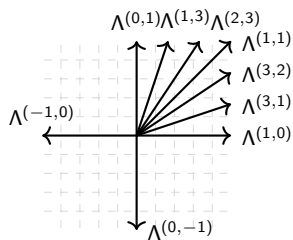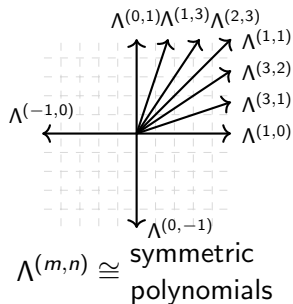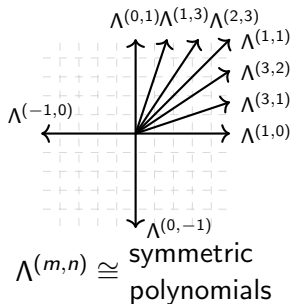$$\mathcal{E} \cong \begin{array}{c}\text{central}\\\text{subalgebra}\end{array} \oplus \bigoplus_{m,n \text{ coprime}} \Lambda^{(m,n)}$$

$\Lambda^{(m,n)} \cong \begin{array}{c}\text{symmetric}\\\text{polynomials}\end{array}$

LHS of Shuffle Theorem $= e_k^{(1,1)} \in \Lambda^{(1,1)}$ acting on $1 \in \Lambda$.
LHS of Rational Shuffle Theorem $= e_k^{(m,n)} \in \Lambda^{(m,n)}$ acting on $1 \in \Lambda$.

Can be difficult to work with in general. Can we make it more explicit?

Root ideals

$R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$ denotes the set of positive roots for GL_n, where $\alpha_{ij} = \epsilon_i - \epsilon_j$.

Root ideals

$R_+ = \{\alpha_{ij} \mid 1 \le i < j \le n\}$ denotes the set of positive roots for GL_n, where $\alpha_{ij} = \epsilon_i - \epsilon_j$.

A root ideal $\Psi \subseteq R_+$ is an upper order ideal of positive roots.

$\Psi =$ Roots above Dyck path

Schur functions revisited

- Convention: $h_0 = 1$ and $h_d = 0$ for $d < 0$.
- For any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$, set

$$s_\gamma = \det(h_{\gamma_i + j - i})_{1 \le i, j \le n}$$

Schur functions revisited

- Convention: $h_0 = 1$ and $h_d = 0$ for $d < 0$.
- For any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$, set

$$s_\gamma = \det(h_{\gamma_i + j - i})_{1 \leq i, j \leq n}$$

Then, $s_\gamma = \pm s_\lambda$ or 0 for some partition λ.

Schur functions revisited

- Convention: $h_0 = 1$ and $h_d = 0$ for $d < 0$.
- For any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$, set

$$s_\gamma = \det(h_{\gamma_i + j - i})_{1 \leq i,j \leq n}$$

Then, $s_\gamma = \pm s_\lambda$ or 0 for some partition λ.
Precisely, for $\rho = (n-1, n-2, \ldots, 1, 0)$,

$$s_\gamma = \begin{cases} \operatorname{sgn}(\gamma + \rho) s_{\operatorname{sort}(\gamma+\rho)-\rho} & \text{if } \gamma + \rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

- $\operatorname{sort}(\beta)$ = weakly decreasing sequence obtained by sorting β,
- $\operatorname{sgn}(\beta)$ = sign of the shortest permutation taking β to $\operatorname{sort}(\beta)$.

Example: $s_{201} = 0$, $s_{2\text{-}11} = -s_{200}$.

Define the *Weyl symmetrization operator* $\boldsymbol{\sigma} \colon \mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \to \Lambda(X)$ by linearly extending

$$\boldsymbol{z}^{\gamma} \mapsto s_{\gamma}(X)$$

where $\boldsymbol{z}^{\gamma} = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$.

Weyl symmetrization

Define the *Weyl symmetrization operator* $\boldsymbol{\sigma} \colon \mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \to \Lambda(X)$ by linearly extending

$$\boldsymbol{z}^{\gamma} \mapsto s_{\gamma}(X)$$

where $\boldsymbol{z}^{\gamma} = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$.

Example

$\boldsymbol{\sigma}(\boldsymbol{z}^{111} + \boldsymbol{z}^{201} + \boldsymbol{z}^{210} + \boldsymbol{z}^{3\text{-}11}) = s_{111} + s_{201} + s_{210} + s_{3\text{-}11} = s_{111} + s_{210} - s_{300}$

Catalanimals

Definition

The *Catalanimal* indexed by $R_q, R_t, R_{qt} \subseteq R_+$ and $\lambda \in \mathbb{Z}^n$ is

Catalanimals

Definition

The *Catalanimal* indexed by $R_q, R_t, R_{qt} \subseteq R_+$ and $\lambda \in \mathbb{Z}^n$ is

$$H(R_q, R_t, R_{qt}, \lambda) = \sigma\left(\frac{\mathbf{z}^\lambda \prod_{\alpha \in R_{qt}}\left(1 - qt\mathbf{z}^\alpha\right)}{\prod_{\alpha \in R_q}\left(1 - q\mathbf{z}^\alpha\right)\prod_{\alpha \in R_t}\left(1 - t\mathbf{z}^\alpha\right)}\right),$$

where $\mathbf{z}^{\alpha_{ij}} = z_i/z_j$ and $(1 - tz_i/z_j)^{-1} = 1 + tz_i/z_j + t^2 z_i^2/z_j^2 + \cdots$.

Catalanimals

Definition

The *Catalanimal* indexed by $R_q, R_t, R_{qt} \subseteq R_+$ and $\lambda \in \mathbb{Z}^n$ is

$$H(R_q, R_t, R_{qt}, \lambda) = \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^\lambda \prod_{\alpha \in R_{qt}} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_q} \left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha \in R_t} \left(1 - t\boldsymbol{z}^\alpha\right)} \right),$$

where $\boldsymbol{z}^{\alpha_{ij}} = z_i/z_j$ and $(1 - tz_i/z_j)^{-1} = 1 + tz_i/z_j + t^2 z_i^2/z_j^2 + \cdots$.

With $n = 3$,

$$H(R_+, R_+, \{\alpha_{13}\}, (111)) = \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^{111}(1 - qtz_1/z_3)}{\prod_{1 \le i < j \le 3}(1 - qz_i/z_j)(1 - tz_i/z_j)} \right)$$

$$= s_{111} + (q + t + q^2 + qt + t^2)s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3)s_3$$

$$= \omega \nabla e_3.$$

Why?

Let $R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq l\}$ and $R_+^0 = \{\alpha_{ij} \in R_+ \mid i + 1 < j\}$.

Why?

Let $R_+ = \{\alpha_{ij} \mid 1 \le i < j \le l\}$ and $R_+^0 = \{\alpha_{ij} \in R_+ \mid i+1 < j\}$.
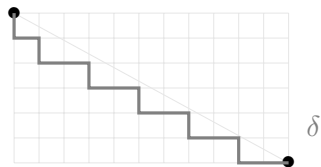
Proposition

For $(m, n) \in \mathbb{Z}_+^2$ coprime,

$$e_k^{(m,n)} \cdot 1 = H(R_+, R_+, R_+^0, \mathbf{b})$$

for $\mathbf{b} = (b_0, \dots, b_{km-1})$ satisfying $b_i =$ the number of south steps on vertical line $x = i$ of highest lattice path under line $y + \frac{n}{m}x = n$.

$\delta =$ highest Dyck path.

δ $\mathbf{b} = (1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0)$

Results

Manipulating Catalanimal $\implies$ a proof of the Rational Shuffle Theorem + a generalization.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $\mathbf{b} = (b_1, \ldots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.

Results

Manipulating Catalanimal $\implies$ a proof of the Rational Shuffle Theorem + a generalization.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $\mathbf{b} = (b_1, \ldots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.

$$H(R_+, R_+, R_+^0, \mathbf{b})$$

Results

Manipulating Catalanimal $\implies$ a proof of the Rational Shuffle Theorem + a generalization.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $\mathbf{b} = (b_1, \ldots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.
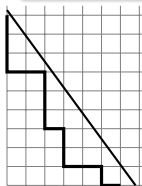
$$H(R_+, R_+, R_+^0, \mathbf{b}) =$$

Results

Manipulating Catalanimal $\implies$ a proof of the Rational Shuffle Theorem + a generalization.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $\mathbf{b} = (b_1, \ldots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.

$$H(R_+, R_+, R_+^0, \mathbf{b}) = \sum_{\lambda} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line $y + px = s$,

Results

Manipulating Catalanimal $\implies$ a proof of the Rational Shuffle Theorem + a generalization.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $\mathbf{b} = (b_1, \ldots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.

$$H(R_+, R_+, R_+^0, \mathbf{b}) = \sum_\lambda t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line $y + px = s$,
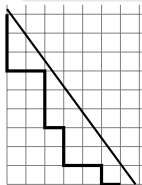
area(λ) as before
$\text{dinv}_p(\lambda) = \#p$-balanced hooks $\frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

Why stop at $e_k^{(m,n)}$?

Why stop at $e_k^{(m,n)}$?

For which symmetric functions f can we find a Catalanimal such that $f^{(m,n)} \cdot 1 =$ a Catalanimal?

Why stop at $e_k^{(m,n)}$?

For which symmetric functions f can we find a Catalanimal such that $f^{(m,n)} \cdot 1 = $ a Catalanimal?

Answer: for f equal to any LLT polynomial!

Why stop at $e_k^{(m,n)}$?

For which symmetric functions f can we find a Catalanimal such that $f^{(m,n)} \cdot 1 =$ a Catalanimal?

Answer: for f equal to any LLT polynomial!

Special case: $\mathcal{G}_\nu^{(1,1)} \cdot 1 = \nabla \mathcal{G}_\nu(X; q)$.

For a tuple of skew shapes ν, the *LLT Catalanimal* $H_{\nu} = H(R_q, R_t, R_{qt}, \lambda)$ is determined by

- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$,

LLT Catalanimals

For a tuple of skew shapes ν, the *LLT Catalanimal* $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$ is determined by

- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$,
- $R_+ \setminus R_q =$ pairs of boxes in the same diagonal in the same shape,
- $R_q \setminus R_t =$ the attacking pairs,
- $R_t \setminus R_{qt} =$ pairs going between adjacent diagonals,

For a tuple of skew shapes $\boldsymbol{\nu}$, the *LLT Catalanimal* $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ is determined by
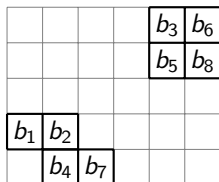
- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$,
- $R_+ \setminus R_q = $ pairs of boxes in the same diagonal in the same shape,
- $R_q \setminus R_t = $ the attacking pairs,
- $R_t \setminus R_{qt} = $ pairs going between adjacent diagonals,
- λ: fill each diagonal D of $\boldsymbol{\nu}$ with
  $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.
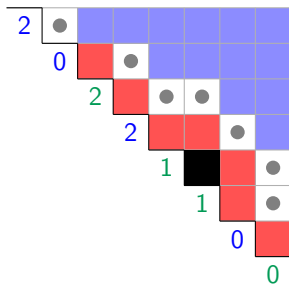 Listing this filling in reading order gives λ.

LLT Catalanimals

$R_+ \setminus R_q =$ pairs of boxes in the same diagonal,

$R_q \setminus R_t =$ the attacking pairs,

$R_t \setminus R_{qt} =$ pairs going between adjacent diagonals,

$R_{qt} =$ all other pairs,

λ: fill each diagonal D of ν with

$1 + \chi(D$ contains a row start$) - \chi(D$ contains a row end$)$.
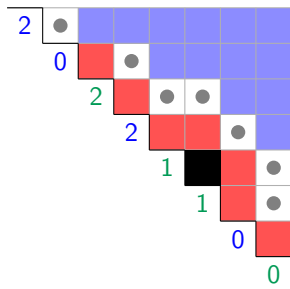

ν

LLT Catalanimals

■ $R_+ \setminus R_q$ = pairs of boxes in the same diagonal,

■ $R_q \setminus R_t$ = the attacking pairs,

• $R_t \setminus R_{qt}$ = pairs going between adjacent diagonals,

■ R_{qt} = all other pairs,

λ: fill each diagonal D of $\boldsymbol{\nu}$ with

$1 + \chi(D$ contains a row start$) - \chi(D$ contains a row end$)$.

λ, as a filling of $\boldsymbol{\nu}$

LLT Catalanimals

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021+)

Let $\boldsymbol{\nu}$ be a tuple of skew shapes and let $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ be the associated LLT Catalanimal. Then

$$\nabla \mathcal{G}_{\boldsymbol{\nu}}(X; q) = c_{\boldsymbol{\nu}} \, \omega H_{\boldsymbol{\nu}}$$

$$= c_{\boldsymbol{\nu}} \, \omega \boldsymbol{\sigma} \left(\frac{\boldsymbol{z}^{\lambda} \prod_{\alpha \in R_{qt}} \left(1 - qt \, \boldsymbol{z}^{\alpha}\right)}{\prod_{\alpha \in R_q} \left(1 - q \, \boldsymbol{z}^{\alpha}\right) \prod_{\alpha \in R_t} \left(1 - t \, \boldsymbol{z}^{\alpha}\right)} \right)$$

for some $c_{\boldsymbol{\nu}} \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$.

- Remember $\nabla \tilde{H}_\mu = q^{n(\mu)} t^{n(\mu^*)} \tilde{H}_\mu$.

- Remember $\nabla \tilde{H}_\mu = q^{n(\mu)} t^{n(\mu^*)} \tilde{H}_\mu$.
- We have a formula for $\nabla \mathcal{G}_{\boldsymbol{\nu}}$.

- Remember $\nabla \tilde{H}_\mu = q^{n(\mu)} t^{n(\mu^*)} \tilde{H}_\mu$.
- We have a formula for $\nabla \mathcal{G}_{\boldsymbol{\nu}}$.
- Does there exist formula $\tilde{H}_\mu = \sum_{\boldsymbol{\nu}} a_{\mu \boldsymbol{\nu}}(q,t) \mathcal{G}_{\boldsymbol{\nu}}$?

- Remember $\nabla \tilde{H}_\mu = q^{n(\mu)} t^{n(\mu^*)} \tilde{H}_\mu$.
- We have a formula for $\nabla \mathcal{G}_{\boldsymbol{\nu}}$.
- Does there exist formula $\tilde{H}_\mu = \sum_{\boldsymbol{\nu}} a_{\mu\boldsymbol{\nu}}(q,t) \mathcal{G}_{\boldsymbol{\nu}}$? Yes!

1. Background on symmetric functions and Macdonald polynomials
2. Shuffle theorems, combinatorics, and LLT polynomials
3. **A new formula for Macdonald polynomials**

$$\tilde{H}_\mu(X; q, t) = \sum_D \left(\prod_{u \in D} q^{-\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1} \right) \mathcal{G}_{\boldsymbol{\nu}(\mu, D)}(X; q)$$
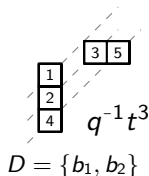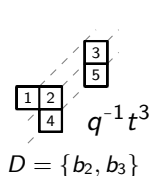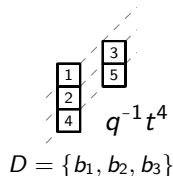
Haglund-Haiman-Loehr formula example

$$\tilde{H}_\mu(X; q, t) = \sum_D \left(\prod_{u \in D} q^{-\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1} \right) \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}(X; q)$$

μ

$q^{-1}t^4$

$D = \{b_1, b_2, b_3\}$

$q^{-1}t^3$

$D = \{b_2, b_3\}$

$q^{-1}t^3$

$D = \{b_1, b_2\}$

t^2

$D = \{b_1, b_3\}$

$q^{-1}t^2$

$D = \{b_2\}$

t

$D = \{b_3\}$

t

$D = \{b_1\}$

1

$D = \varnothing$

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}$ and apply $\omega \nabla$.

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}$ and apply $\omega\nabla$.
- By construction, all the LLT Catalanimals $H_{\boldsymbol{\nu}(\mu,D)}$ appearing on the RHS will have the same root ideal data (R_q, R_t, R_{qt}).

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}$ and apply $\omega \nabla$.
- By construction, all the LLT Catalanimals $H_{\boldsymbol{\nu}(\mu,D)}$ appearing on the RHS will have the same root ideal data (R_q, R_t, R_{qt}).
- Collect terms to get $\prod_{(b_i,b_j) \in V(\mu)} (1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j)$ factor for $V(\mu)$ the set of vertical dominoes (b_i, b_j) in μ.

$$\tilde{H}_\mu = \omega \boldsymbol{\sigma} \left(z_1 \cdots z_n \frac{\displaystyle\prod_{\alpha_{ij} \in V(\mu)} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod_{\alpha \in \widehat{R}_\mu} \left(1 - qt\mathbf{z}^\alpha\right)}{\prod_{\alpha \in R_+} \left(1 - q\mathbf{z}^\alpha\right) \prod_{\alpha \in R_\mu} \left(1 - t\mathbf{z}^\alpha\right)} \right).$$

row reading order
$b_1 \prec b_2 \prec \cdots \prec b_n$

$R_\mu := \left\{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \preceq b_j \right\},$

$\widehat{R}_\mu := \left\{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \prec b_j \right\},$

$R_\mu \setminus \widehat{R}_\mu \leftrightarrow V(\mu) = \text{vertical dominoes in } \mu$

Example:

$R_{3321} =$

The root ideal R_μ

row reading order
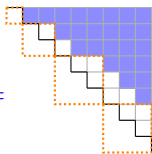$$b_1 \prec b_2 \prec \cdots \prec b_n$$

$$R_\mu := \big\{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \preceq b_j \big\},$$

$$\widehat{R}_\mu := \big\{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \prec b_j \big\},$$

$$R_\mu \setminus \widehat{R}_\mu \leftrightarrow V(\mu) = \text{vertical dominoes in } \mu$$

Example:

$$R_{3321} = $$
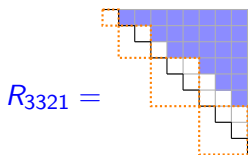
Remark

$$\tilde{H}_\mu(X; 0, t) = \omega \boldsymbol{\sigma} \left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu} (1 - t\boldsymbol{z}^\alpha)} \right)$$

Example
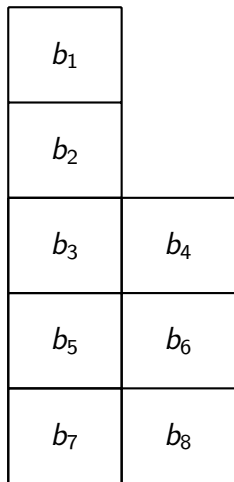
partition $\mu = 22211$
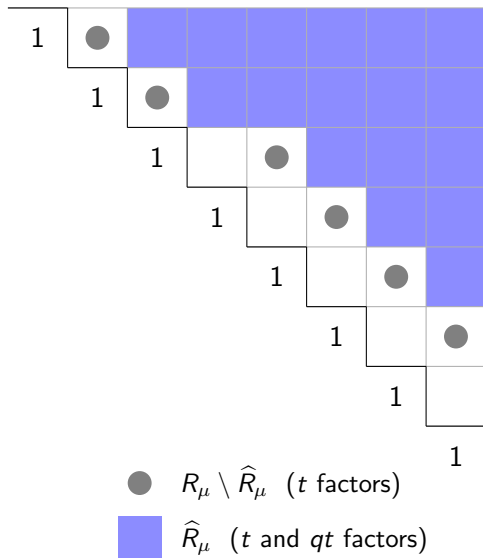
$\bullet$ $R_\mu \setminus \widehat{R}_\mu$ (t factors)

$\widehat{R}_\mu$ (t and qt factors)

Example

numerator factors $1 - q^{\mathrm{arm}+1} t^{-\mathrm{leg}} z_i/z_j$

$\tilde{H}_{22211}$
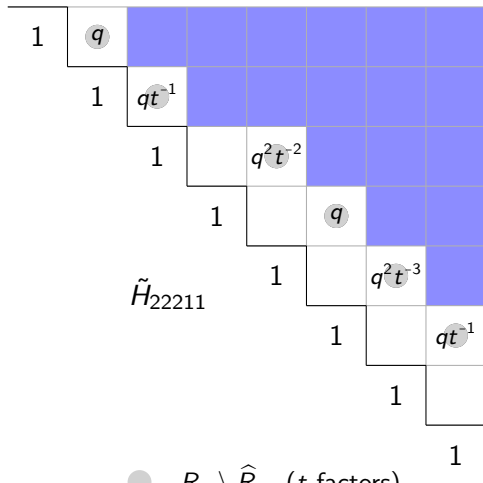
● $R_\mu \setminus \widehat{R}_\mu$ (t factors)

■ $\widehat{R}_\mu$ (t and qt factors)

$q = t = 1$ specialization

$$\omega\boldsymbol{\sigma}\left(z_1\cdots z_n\frac{\displaystyle\prod_{\alpha_{ij}\in R_\mu\setminus\widehat{R}_\mu}\left(1-q^{\mathrm{arm}(b_i)+1}t^{-\mathrm{leg}(b_i)}z_i/z_j\right)\prod_{\alpha\in\widehat{R}_\mu}\left(1-qtz^\alpha\right)}{\prod_{\alpha\in R_+}\left(1-qz^\alpha\right)\prod_{\alpha\in R_\mu}\left(1-tz^\alpha\right)}\right)$$

$$\overset{q=t=1}{\to}\omega\boldsymbol{\sigma}\left(z_1\cdots z_n\frac{\prod_{\alpha\in R_\mu\setminus\widehat{R}_\mu}(1-\boldsymbol{z}^\alpha)\prod_{\alpha\in\widehat{R}_\mu}(1-\boldsymbol{z}^\alpha)}{\prod_{\alpha\in R_+}(1-\boldsymbol{z}^\alpha)\prod_{\alpha\in R_\mu}(1-\boldsymbol{z}^\alpha)}\right)$$

$$=\omega\boldsymbol{\sigma}\left(\frac{z_1\cdots z_n}{\prod_{\alpha\in R_+}(1-\boldsymbol{z}^\alpha)}\right)$$

$$=\omega h_1^n$$

$$=e_1^n$$

A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

$$\tilde{H}_\mu^{(s)} := \omega\boldsymbol{\sigma}\left((z_1\cdots z_n)^s \frac{\prod\limits_{\alpha_{ij}\in R_\mu\setminus\widehat{R}_\mu}\left(1 - q^{\mathrm{arm}(b_i)+1}t^{-\mathrm{leg}(b_i)}z_i/z_j\right)\prod\limits_{\alpha\in\widehat{R}_\mu}\left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+}\left(1 - q\boldsymbol{z}^\alpha\right)\prod_{\alpha\in R_\mu}\left(1 - t\boldsymbol{z}^\alpha\right)}\right)$$

Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition μ and positive integer s, the symmetric function $\tilde{H}_\mu^{(s)}$ is Schur positive. That is, the coefficients in

$$\tilde{H}_\mu^{(s)} = \sum_\nu K_{\nu,\mu}^{(s)}(q,t)\, s_\nu(X)$$

satisfy $K_{\nu,\mu}^{(s)}(q,t)\in\mathbb{N}[q,t]$.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	$SSYT(\lambda)$
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	HHL
∇e_n	DH_n	Shuffle theorem
$\tilde{H}_\lambda^{(s)}(X; q, t)$	??	??

Thank you!

Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. 2023/ed. *A Shuffle Theorem for Paths Under Any Line*, Forum of Mathematics, Pi **11**, e5, DOI 10.1017/fmp.2023.4.

———. 2021. *LLT Polynomials in the Schiffmann Algebra*, arXiv e-prints, arXiv:2112.07063.

———. 2023. *A Raising Operator Formula for Macdonald Polynomials*, arXiv e-prints, arXiv:2307.06517.

Burban, Igor and Olivier Schiffmann. 2012. *On the Hall algebra of an elliptic curve, I*, Duke Math. J. **161**, no. 7, 1171–1231, DOI 10.1215/00127094-1593263. MR2922373

Carlsson, Erik and Mellit, Anton. 2018. *A Proof of the Shuffle Conjecture* **31**, no. 3, 661–697, DOI 10.1090/jams/893.

Feigin, B. L. and Tsymbaliuk, A. I. 2011. *Equivariant K-theory of Hilbert Schemes via Shuffle Algebra*, Kyoto J. Math. **51**, no. 4, 831–854.

Garsia, Adriano M. and Mark Haiman. 1993. *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A. **90**, no. 8, 3607–3610, DOI 10.1073/pnas.90.8.3607. MR1214091

Haglund, J., M. Haiman, and N. Loehr. 2005. *A Combinatorial Formula for Macdonald Polynomials* **18**, no. 3, 735–761 (electronic).

Haglund, J. and Haiman, M. and Loehr. 2005. *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J. **126**, no. 2, 195–232, DOI 10.1215/S0012-7094-04-12621-1.

Haiman, Mark. 2001. *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14**, no. 4, 941–1006, DOI 10.1090/S0894-0347-01-00373-3. MR1839919

———. 2002. *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149**, no. 2, 371–407, DOI 10.1007/s002220200219. MR1918676

Lascoux, Alain, Bernard Leclerc, and Jean-Yves Thibon. 1995. *Ribbon tableaux, Hall-Littlewood functions and unipotent varieties*, Sém. Lothar. Combin. **34**, Art. B34g, approx. 23. MR1399754

Mellit, Anton. 2021. *Toric Braids and (m,n)-Parking Functions*, Duke Math. J. **170**, no. 18, 4123–4169, DOI 10.1215/00127094-2021-0011.

Negut, Andrei. 2014. *The shuffle algebra revisited*, Int. Math. Res. Not. IMRN **22**, 6242–6275, DOI 10.1093/imrn/rnt156. MR3283004

Schiffmann, Olivier and Vasserot, Eric. 2013. *The Elliptic Hall Algebra and the K-theory of the Hilbert Scheme of A2*, Duke Mathematical Journal **162**, no. 2, 279–366, DOI 10.1215/00127094-1961849.