

Diagonal Harmonics and Shuffle Theorems

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on joint work with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun
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OIST Representation Theory Seminar

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- Symmetric polynomials and diagonal harmonics
- The Shuffle Theorem and its generalizations
- Proof techniques and new progress

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

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Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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- E.g. for $n = 3$,

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

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- Λ is a $\mathbb{Q}(q, t)$ -algebra.

Distinguished basis of Schur polynomials

$$s_{\mu}(x_1, \dots, x_l) = \sum_{w \in S_l} w \left(\frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i < j} (1 - x_j/x_i)} \right)$$

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- $s_{\lambda} = \sum_{\mu} \frac{\chi_{\lambda}(\mu)}{z_{\mu}} p_{\mu}$ for irreducible S_n -character χ_{λ} .

Schur Polynomials

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Hidden Guide: Schur Positivity

“Naturally occurring” symmetric functions which are non-negative (coefficients in $\mathbb{N}[q, t]$) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

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$$\begin{aligned} M &= \text{sp} \left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

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Remark: $M \cong \mathbb{C}[x_1, x_2, x_3]/(\mathbb{C}[x_1, x_2, x_3]_{+}^{S_3})$.

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Solution: irreducible S_n -representation of polynomials of degree $d \mapsto q^d s_\lambda$
(graded Frobenius)

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Answer: Hall-Littlewood polynomial $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$.

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- Does there exist a family of S_n -regular representations whose bigraded Frobenius characteristics equal $\tilde{H}_\lambda(X; q, t)$?

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Irreducible S_n -representation with bidegree $(a, b) \mapsto q^a t^b s_\lambda$

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Irreducible S_n -representation with bidegree $(a, b) \mapsto q^a t^b s_\lambda$

$$\tilde{H}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} = qts \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} + ts \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + qs \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + s \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}$$

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- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$.

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left(\sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

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Question

What symmetric function is the bigraded Frobenius characteristic of DH_n ?

Frobenius characteristic of DH_3

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$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

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Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n .

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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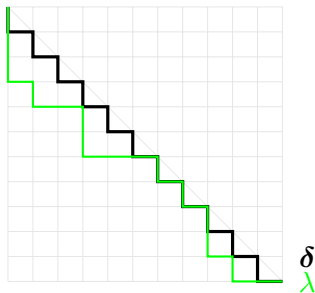
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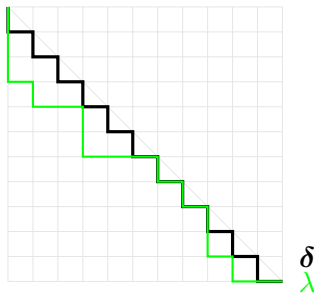
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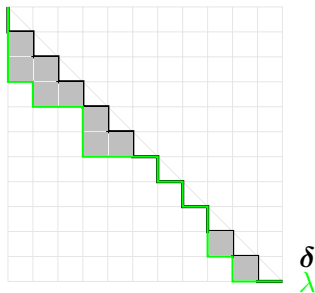
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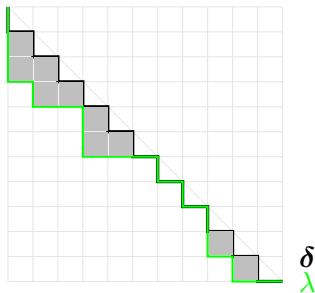
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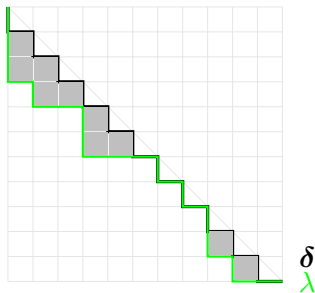
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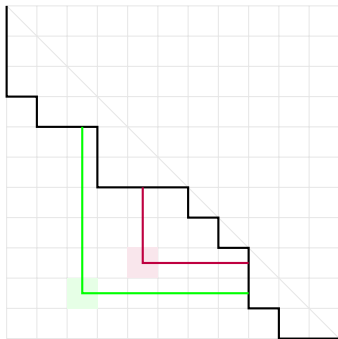
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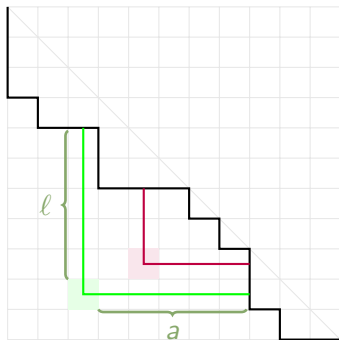
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dinv

$\text{dinv}(\lambda) = \#$ of balanced hooks in diagram below λ .



$\text{divv}(\lambda) = \#$ of balanced hooks in diagram below λ .



Balanced hook is given by a cell below λ satisfying

$$\frac{l}{a+1} < 1 - \epsilon < \frac{l+1}{a}, \quad \epsilon \text{ small.}$$

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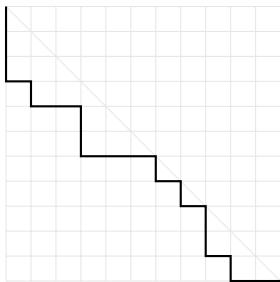
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- \mathcal{G}_ν is Schur-positive for any tuple of skew shapes ν [Grojnowski-Haiman, 2007].

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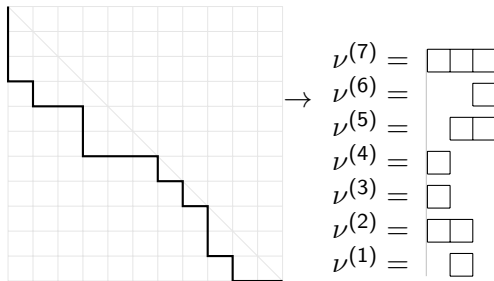
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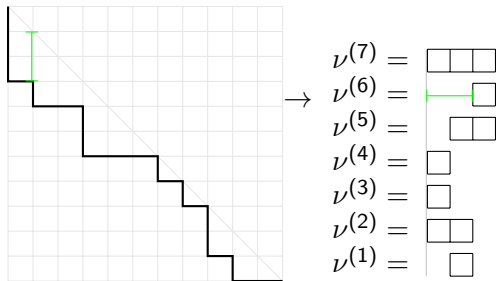
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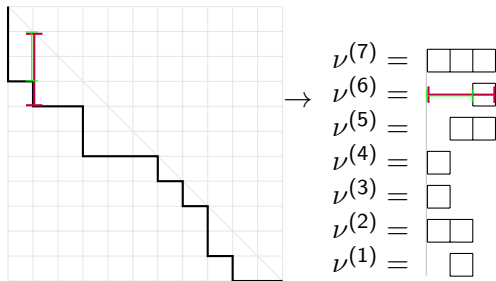
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1 2 3 3 5

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$$= s_3 + q s_{2,1}$$

Example ∇e_3






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




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- Symmetric polynomials and diagonal harmonics
- **The Shuffle Theorem and its generalizations**
- Proof techniques and new progress

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- Burban and Schiffmann studied a subalgebra \mathcal{E} of the Hall algebra of coherent sheaves on an elliptic curve over \mathbb{F}_p

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- Burban and Schiffmann studied a subalgebra \mathcal{E} of the Hall algebra of coherent sheaves on an elliptic curve over \mathbb{F}_p
- \mathcal{E} contains, for every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$, with relations between them. (Burban-Schiffmann, 2012)

- \mathcal{E} acts on Λ , e.g., for $M = (1 - q)(1 - t)$ and automorphism ω ,

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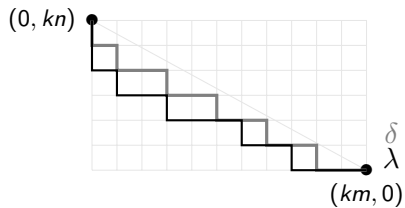
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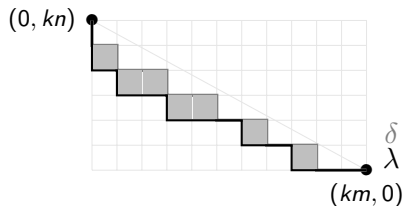
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- Coefficient of $s_{1, \dots, 1}$ is “rational (q, t) -Catalan number”

Rational Path Combinatorics

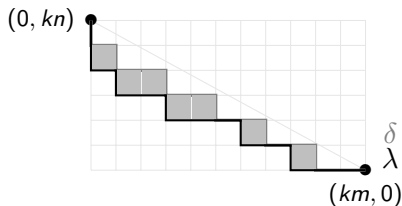


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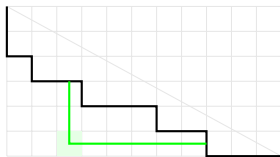


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$$\frac{\ell}{a+1} < p < \frac{\ell+1}{a} \quad p = \frac{n}{m} - \epsilon$$

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For $\mathbf{b} \in \mathbb{Z}^l$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

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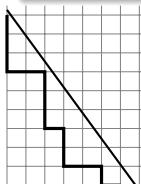
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- Symmetric polynomials and The Shuffle Theorem
- Generalizations of The Shuffle Theorem
- **Proof techniques and new progress**

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$$\phi \mapsto \sum_{w \in S_l} w \left(\phi \prod_{i < j} \frac{(1 - qtz_i/z_j)}{(1 - z_j/z_i)(1 - qz_i/z_j)(1 - tz_i/z_j)} \right).$$

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Key Relationship

$$\omega(D_{\mathbf{b}} \cdot 1)(x_1, \dots, x_l) = \left(\sum_{w \in S_l} w \left(\frac{x_1^{b_1} \cdots x_l^{b_l} \prod_{i+1 < j} (1 - qtx_i/x_{i+1})}{\prod_{i < j} ((1 - \frac{x_j}{x_i})(1 - q\frac{x_i}{x_j})(1 - t\frac{x_i}{x_j}))} \right) \right)_{\text{pol}}$$

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- Need an “infinite series” version of LLT polynomials!

Cauchy Identity

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Proof Idea

$$\text{Let } H_q(f) = \sigma \left(\frac{f}{\prod_{i < j} (1 - qx_i/x_j)} \right).$$

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Convex Curve Conjecture (Blasiak-Haiman-Morse-Pun-S., 2021a)

For $\mathbf{b} = (b_1, \dots, b_l)$ the south steps of highest path under a convex curve, the Schur expansion of $D_{\mathbf{b}} \cdot 1$ has coefficients in $\mathbb{N}[q, t]$.

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- Experimental computation suggests this is “tight.”
- Coefficient of $s_{1, \dots, 1}$ coincides with (q, t) -polynomials found in (Gorsky-Hawkes-Schilling-Rainbolt, 2020), (Galashin-Lam, 2021).

Loehr-Warrington Conjecture (2008)

$$\nabla s_{\mu} = \operatorname{sgn}(\mu) \sum_{(G,R) \in \operatorname{LNDP}_{\mu}} t^{\operatorname{area}(G,R)} q^{\operatorname{dinv}(G,R)} x^R$$

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- S_I -representation theory interpretations?

Thank you!

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