

# A raising operator formula for Macdonald polynomials via LLT polynomials in the Schiffmann algebra

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- ① **Background on symmetric functions and Macdonald polynomials**
- ② A new formula for Macdonald polynomials
- ③ LLT polynomials in the elliptic Hall algebra

# Symmetric Polynomials

- Polynomials  $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$  satisfying  $\sigma.f = f$  for all  $\sigma \in S_n$ .

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## Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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- E.g. for  $n = 3$ ,

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

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- $\Lambda$  is a  $\mathbb{Q}(q, t)$ -algebra.

# Symmetric functions and Schur functions

- Convention:  $h_0 = 1$  and  $h_d = 0$  for  $d < 0$ .
- For any  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n$ ,

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Then, for  $\rho = (n-1, n-2, \dots, 1, 0)$ ,

$$s_\gamma = \begin{cases} \operatorname{sgn}(\gamma + \rho) s_{\operatorname{sort}(\gamma + \rho) - \rho} & \text{if } \gamma + \rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

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Example:  $s_{201} = 0$ ,  $s_{2-11} = -s_{200}$ .

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## Hidden Guide: Schur Positivity

“Naturally occurring” symmetric functions which are non-negative (coefficients in  $\mathbb{N}[q, t]$ ) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

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$$\begin{aligned} M &= \text{sp} \left\{ \left( \partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

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Remark:  $M \cong \mathbb{C}[x_1, x_2, x_3]/(\mathbb{C}[x_1, x_2, x_3]_{+}^{S_3})$ .

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Solution: irreducible  $S_n$ -representation of polynomials of degree  $d \mapsto q^d s_\lambda$   
(graded Frobenius)

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Answer: Hall-Littlewood polynomial  $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$ .

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- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$ .
- Does there exist a family of  $S_n$ -regular representations whose bigraded Frobenius characteristics equal  $\tilde{H}_\lambda(X; q, t)$ ?

## Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .

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Irreducible  $S_n$ -representation with bidegree  $(a, b) \mapsto q^a t^b s_\lambda$

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .
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## Theorem (Haiman, 2001)

*The Garsia-Haiman module  $M_\lambda$  has bigraded Frobenius characteristic given by  $\tilde{H}_\lambda(X; q, t)$*



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- No combinatorial description of  $\tilde{K}_{\lambda\mu}(q, t)$ .

- ① Background on symmetric functions and Macdonald polynomials
- ② **A new formula for Macdonald polynomials**
- ③ LLT polynomials in the elliptic Hall algebra

# Root ideals

$R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$  denotes the set of positive roots for  $GL_n$ , where  $\alpha_{ij} = \epsilon_i - \epsilon_j$ .

	(12)	(13)	(14)	(15)
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A root ideal  $\Psi \subseteq R_+$  is an upper order ideal of positive roots.

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$\Psi =$  Roots above Dyck path

# Weyl symmetrization

Define the *Weyl symmetrization operator*  $\sigma: \mathbb{Q}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \rightarrow \Lambda(X)$  by linearly extending

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$$H(\Phi; \gamma) = \sigma \left( \frac{z^\gamma}{\prod_{(i,j) \in \Psi} (1 - tz_i/z_j)} \right)$$

Denominator factors are understood as geometric series  
 $(1 - tz_i/z_j)^{-1} = 1 + tz_i/z_j + t^2(z_i/z_j)^2 + \cdots$

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Example:

$$\Psi = \begin{array}{c} \blacksquare \\ \square \\ \square \end{array} \quad \gamma = (1, 1, 1)$$

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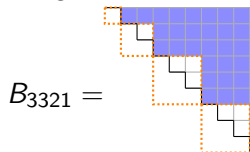
Example:

$$\Psi = \begin{array}{|c|} \hline \square \\ \hline \end{array} \quad \gamma = (1, 1, 1)$$

$$\begin{aligned} H(\Psi; \gamma) &= \sigma \left( \left(1 + t \frac{z_1}{z_2} + t^2 \frac{z_1^2}{z_2^2} + \dots\right) \left(1 + t \frac{z_1}{z_3} + t^2 \frac{z_1^2}{z_3^2} + \dots\right) z_1 z_2 z_3 \right) \\ &= s_{111} + t(s_{201} + s_{210}) + t^2(s_{3-10} + s_{300} + s_{31-1}) + \dots \\ &= s_{111} + ts_{210} \end{aligned}$$

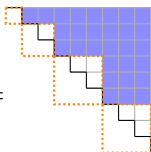
# A Catalan function for modified Hall-Littlewoods

$B_\mu$  = set of roots above block diagonal matrix with block sizes  $\mu_{\ell(\mu)}, \dots, \mu_1$



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Theorem (Weyman, Shimozono-Weyman)

$$\tilde{H}_\mu(X; 0, t) = \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in B_\mu} (1 - t z^\alpha)} \right),$$

where  $z^\alpha = z_i / z_j$ .

$\omega(s_\lambda) = s_{\lambda'}$  for  $\lambda'$  the transpose partition of  $\lambda$ .

# Catalan functions for modified Hall-Littlewoods

$b_1$		
$b_2$	$b_3$	
$b_4$	$b_5$	$b_6$
$b_7$	$b_8$	$b_9$

row reading order

$$b_1 \prec b_2 \prec \cdots \prec b_n$$

$$R_\mu := \{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \preceq b_j \}.$$

$$R_{3321} =$$

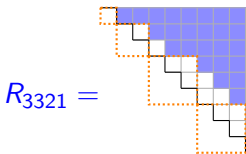
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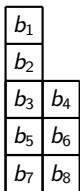
$b_1 \prec b_2 \prec \dots \prec b_n$

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$$\begin{aligned} \tilde{H}_\mu(X; 0, t) &= \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in B_\mu} (1 - tz^\alpha)} \right), \\ &= \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right) \end{aligned}$$

# A formula for $\tilde{H}_\mu(X; q, t)$



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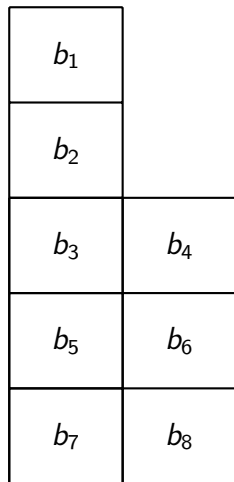
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## Theorem (Blasiak-Haiman-Morse-Pun-S.)

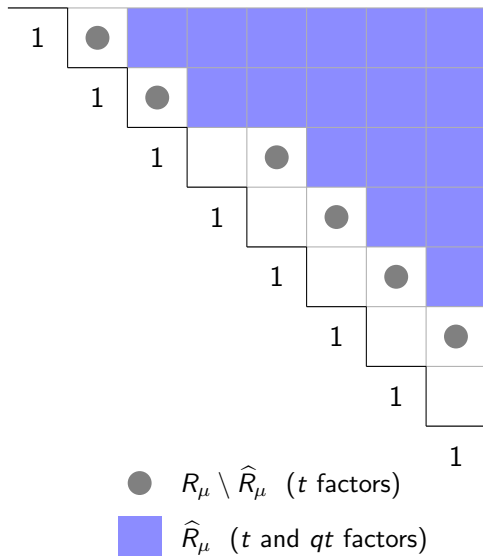
The modified Macdonald polynomial  $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$  is given by

$$\tilde{H}_\mu = \omega \sigma \left( z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qt z^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right).$$

# Example



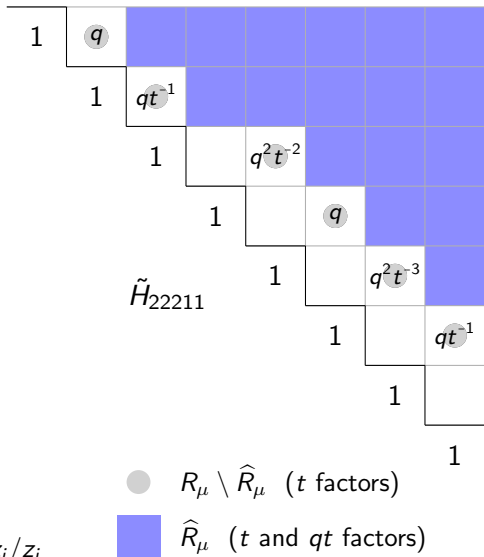
partition  $\mu = 22211$



# Example

$1 - q \frac{z_1}{z_2}$	
$1 - qt^{-1} \frac{z_2}{z_3}$	
$1 - q^2 t^{-2} \frac{z_3}{z_5}$	$1 - q \frac{z_4}{z_6}$
$1 - q^2 t^{-3} \frac{z_5}{z_7}$	$1 - qt^{-1} \frac{z_6}{z_8}$

numerator factors  $1 - q^{\text{arm}+1} t^{-\text{leg}} z_i/z_j$



# $q = t = 1$ specialization

$$\begin{aligned} & \omega\sigma \left( z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j) \prod_{\alpha \in \widehat{R}_\mu} (1 - qtz^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right) \\ \xrightarrow{q=t=1} & \omega\sigma \left( z_1 \cdots z_n \frac{\prod_{\alpha \in R_\mu \setminus \widehat{R}_\mu} (1 - z^\alpha) \prod_{\alpha \in \widehat{R}_\mu} (1 - z^\alpha)}{\prod_{\alpha \in R_+} (1 - z^\alpha) \prod_{\alpha \in R_\mu} (1 - z^\alpha)} \right) \\ & = \omega\sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in R_+} (1 - z^\alpha)} \right) \\ & = \omega h_1^n \\ & = e_1^n \end{aligned}$$

# $q = 0$ specialization

$$\begin{aligned} & \omega\sigma \left( z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j) \prod_{\alpha \in \widehat{R}_\mu} (1 - qtz^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right) \\ & \xrightarrow{q=0} \omega\sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right) \\ & = \widetilde{H}_\mu(X; 0, t) \end{aligned}$$

# Proof of formula for $\tilde{H}_\mu$

## Definition

$\nabla$  is the linear operator on symmetric functions satisfying  
 $\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu$ , where  $n(\mu) = \sum_i (i-1)\mu_i$ .

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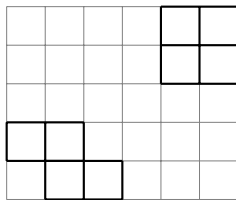
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- Apply  $\omega \nabla$  to both sides.
- Use Catalan-like (“Catalanimal”) formula for  $\omega \nabla \mathcal{G}_\nu(X; q)$  and collect terms.

# LLT Polynomials

Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes.

$$\nu = \left( \begin{array}{|c|c|c|} \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right)$$



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-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

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- A pair  $(a, b) \in \nu$  is *attacking* if  $a$  precedes  $b$  in reading order and
  - $\text{content}(b) = \text{content}(a)$ , or
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Attacking pairs:  $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

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				$b_5$	$b_8$
$b_1$	$b_2$				
	$b_4$	$b_7$			

Attacking pairs:  $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

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- A *semistandard tableau* on  $\nu$  is a map  $T: \nu \rightarrow \mathbb{Z}_+$  which restricts to a semistandard tableau on each  $\nu_{(i)}$ .
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The *LLT polynomial* indexed by a tuple of skew shapes  $\nu$  is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} \mathbf{x}^T,$$

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2	4				
	3	5			

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inversion

$$\text{inv}(T) = 4, \quad \mathbf{x}^T = x_1^2 x_2 x_3 x_4 x_5^2 x_6$$

The *Catalanimal* indexed by  $R_q, R_t, R_{qt} \subseteq R_+$  and  $\lambda \in \mathbb{Z}^n$  is

$$H(R_q, R_t, R_{qt}, \lambda) = \sigma \left( \frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qtz^\alpha)}{\prod_{\alpha \in R_q} (1 - qz^\alpha) \prod_{\alpha \in R_t} (1 - tz^\alpha)} \right).$$

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With  $n = 3$ ,

$$\begin{aligned} H(R_+, R_+, \{\alpha_{13}\}, (111)) &= \sigma \left( \frac{z^{111} (1 - qtz_1/z_3)}{\prod_{1 \leq i < j \leq 3} (1 - qz_i/z_j) (1 - tz_i/z_j)} \right) \\ &= s_{111} + (q + t + q^2 + qt + t^2) s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3) s_3 \\ &= \omega \nabla e_3. \end{aligned}$$

# LLT Catalanimals

For a tuple of skew shapes  $\nu$ , the *LLT Catalanimal*  $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$  is determined by

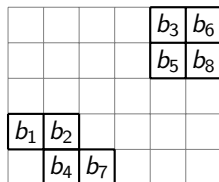
- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$ ,
- $R_+ \setminus R_q =$  pairs of boxes in the same diagonal,
- $R_q \setminus R_t =$  the attacking pairs,
- $R_t \setminus R_{qt} =$  pairs going between adjacent diagonals,
- $\lambda$ : fill each diagonal  $D$  of  $\nu$  with  $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$ .  
Listing this filling in reading order gives  $\lambda$ .

# LLT Catalanimals

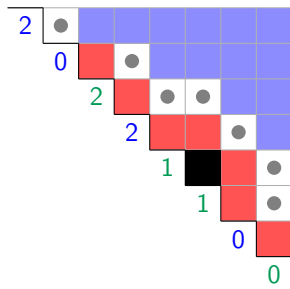
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$1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$ .



$\nu$





# LLT Catalanimals

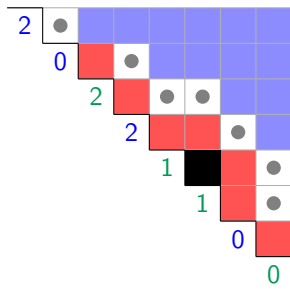
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$\lambda$ : fill each diagonal  $D$  of  $\nu$  with

$1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$ .

				2	1
				1	0
2	0				
	2	0			

$\lambda$ , as a filling of  $\nu$



## Theorem (Blasiak-Haiman-Morse-Pun-S.)

Let  $\nu$  be a tuple of skew shapes and let  $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$  be the associated LLT Catalanimal. Then

$$\begin{aligned}\nabla \mathcal{G}_\nu(X; q) &= c_\nu \omega H_\nu \\ &= c_\nu \omega \sigma \left( \frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qt z^\alpha)}{\prod_{\alpha \in R_q} (1 - q z^\alpha) \prod_{\alpha \in R_t} (1 - t z^\alpha)} \right)\end{aligned}$$

for some  $c_\nu \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$ .

# Haglund-Haiman-Loehr formula

Theorem (Haglund-Haiman-Loehr, 2005)

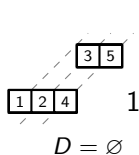
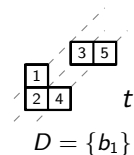
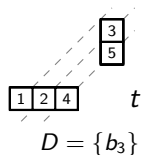
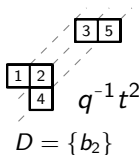
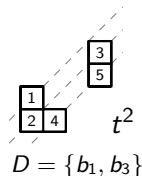
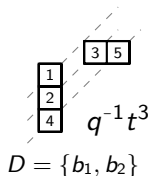
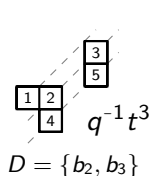
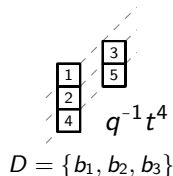
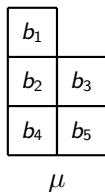
$$\tilde{H}_\mu(X; q, t) = \sum_D \left( \prod_{u \in D} q^{-\text{arm}(u)} t^{\text{leg}(u)+1} \right) \mathcal{G}_{\nu(\mu, D)}(X; q),$$

where

- the sum runs over all subsets  $D \subseteq \{(i, j) \in \mu \mid j > 1\}$ , and
- $\nu(\mu, D) = (\nu^{(1)}, \dots, \nu^{(k)})$  where  $k = \mu_1$  is the number of columns of  $\mu$ , and  $\nu^{(i)}$  is a ribbon of size  $\mu_i^*$ , i.e., box contents  $\{-1, -2, \dots, -\mu_i^*\}$ , and descent set  $\text{Des}(\nu^{(i)}) = \{-j \mid (i, j) \in D\}$ .

# Haglund-Haiman-Loehr formula example

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## Putting it all together

- Take HHL formula  $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_\nu(\mu,D)$  and apply  $\omega \nabla$ .

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- Collect terms to get  $\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j)$  factor.

$$\tilde{H}_\mu = \omega \sigma \left( z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qt z^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right).$$

- ① Background on symmetric functions and Macdonald polynomials
- ② A new formula for Macdonald polynomials
- ③ **LLT polynomials in the elliptic Hall algebra**



Burban and Schiffmann studied a subalgebra  $\mathcal{E}$  of the Hall algebra of coherent sheaves on an elliptic curve over  $\mathbb{F}_p$ .

The *elliptic Hall algebra*  $\mathcal{E}$  is generated by subalgebras  $\Lambda(X^{a,b})$  isomorphic to the ring of symmetric functions  $\Lambda$  over  $\mathbb{k} = \mathbb{Q}(q, t)$ , one for each coprime pair  $(a, b) \in \mathbb{Z}^2$ , along with an additional central subalgebra.

# Shuffle algebra

Define a linear map

$$\sigma_{\Gamma}: \bigoplus_n \mathbb{k}(z_1, \dots, z_n) \rightarrow \bigoplus_n \mathbb{k}(z_1, \dots, z_n)^{S_n}$$

whose graded components  $\sigma_{\Gamma}^n$  are given by

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The *shuffle algebra*  $\mathcal{S}_{\Gamma}$  is the image of  $\bigoplus_n \mathbb{k}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$  under the map  $\sigma_{\Gamma}$ , equipped with a variant of the concatenation product.

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Nice fact (up to some modifications of definitions)

Some Catalan animals are elements in  $\mathcal{S}_{\Gamma}$ . (“Tame Catalan animals”)

# Shuffle to elliptic Hall isomorphism

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- $\mathcal{S}_\Gamma = \sigma_\Gamma \left( \bigoplus_n \mathbb{k}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \right)$  ( $\Gamma$ -symmetrized Laurent polynomials).

# Shuffle to elliptic Hall isomorphism

- The *right half-plane subalgebra*  $\mathcal{E}^+ \subseteq \mathcal{E}$  is generated by  $\Lambda(X^{a,b})$  for  $a > 0$ .
- $\mathcal{S}_\Gamma = \sigma_\Gamma \left( \bigoplus_n \mathbb{k}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \right)$  ( $\Gamma$ -symmetrized Laurent polynomials).

## Theorem (Schiffmann-Vasserot)

There is an algebra isomorphism  $\psi: \mathcal{S}_\Gamma \rightarrow \mathcal{E}^+$ .



## Elliptic Hall algebra action

Schiffmann-Vasserot and Feigin-Tsybaliuk constructed an action of  $\mathcal{E}$  on  $\Lambda$ , where  $f(X^{0,1})$  acts by multiplication by  $f(X)$ .

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### Proposition

*Conjugation by  $\nabla$  provides a symmetry of the action of  $\mathcal{E}$  on  $\Lambda$ ,*

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### Theorem (Blasiak-Haiman-Morse-Pun-S.)

*Let  $H$  be a Catalan animal such that  $\psi(H) = f(X^{1,1})$ . Then*

$$\nabla f = \omega H.$$

# Shuffle to elliptic Hall summary

$$\begin{array}{ccc}
 & & \mathcal{E} \curvearrowright \Lambda \\
 & & \uparrow \\
 \bigoplus_{\substack{a>0 \\ b \in \mathbb{Z} \\ (a,b)=1}} \Lambda(X^{a,b}) & \cong_{\text{v.sp.}} & \mathcal{E}^+ \\
 & & \uparrow \cong \\
 \sigma_\Gamma \left( \bigoplus_n \mathbb{k}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \right) & \cong_{\text{v.sp.}} & \mathcal{S}_\Gamma \ni H \quad \text{“tame” Catalanimal}
 \end{array}$$

Theorem (Blasiak-Haiman-Morse-Pun-S.)

$$\psi(H) = f(X^{1,1}) \implies f(X^{1,1}) \cdot 1 = \nabla f = \omega H.$$

# Proof of $\nabla\mathcal{G}_\nu$ formula

- 1 LLT Catalananimals  $H_\nu$  are tame.
- 2 LLT Catalananimals lie in  $\psi^{-1}(\Lambda(X^{1,1}))$ .
- 3 Describe coproduct  $\Delta$  on  $\mathcal{E}$  explicitly on tame Catalananimals and show  $\Delta H_\nu$  matches  $\Delta\mathcal{G}_\nu$ .
- 4 Conclude  $\psi(H_\nu) = c_\nu^{-1}\mathcal{G}_\nu(X^{1,1}) \in \mathcal{E}$ .
- 5 Apply previous theorem to conclude  $\nabla\mathcal{G}_\nu = c_\nu\omega H_\nu$

## A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

# A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

$$\tilde{H}_\mu^{(s)} := \omega \sigma \left( (z_1 \cdots z_n)^s \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qtz^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right)$$

## Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition  $\mu$  and positive integer  $s$ , the symmetric function  $\tilde{H}_\mu^{(s)}$  is Schur positive. That is, the coefficients in

$$\tilde{H}_\mu^{(s)} = \sum_{\nu} K_{\nu, \mu}^{(s)}(q, t) s_\nu(X)$$

satisfy  $K_{\nu, \mu}^{(s)}(q, t) \in \mathbb{N}[q, t]$ .



# Thank you!

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# Catalan animals in the shuffle algebra

For  $\lambda \in \mathbb{Z}^n$ ,

$$\begin{aligned}\sigma_{\Gamma}^n(\mathbf{z}^{\lambda}) &= \sum_{w \in \mathcal{S}_n} w \left( \frac{\mathbf{z}^{\lambda} \prod_{\alpha \in R_+} (1 - qt\mathbf{z}^{\alpha})}{\prod_{\alpha \in R_+} ((1 - \mathbf{z}^{-\alpha})(1 - q\mathbf{z}^{\alpha})(1 - t\mathbf{z}^{\alpha}))} \right) \\ &= H(R_+, R_+, R_+, \lambda) \in \mathcal{S}_{\Gamma}.\end{aligned}$$

- Technicality: we have redefined

$$\sigma(\mathbf{z}^{\gamma}) = \sum_{w \in \mathcal{S}_n} \left( \frac{\mathbf{z}^{\gamma}}{\prod_{\alpha \in R_+} (1 - \mathbf{z}^{-\alpha})} \right) = \chi_{\gamma}, \text{ the irreducible } \mathrm{GL}_n \text{ character.}$$

- Let  $\mathrm{pol}_X$  send  $\chi_{\lambda} \mapsto s_{\lambda}$  if  $\lambda_n \geq 0$ , otherwise  $\chi_{\lambda} \mapsto 0$ .
- The  $\sigma$  from before is given by  $\sigma_{\mathrm{old}} = \mathrm{pol}_X \sigma_{\mathrm{new}}$ .

# Catalanimals in the Shuffle algebra

$\sigma_{\Gamma}^n(f)$  can lie in  $\mathcal{S}_{\Gamma}$  even when  $f$  is not a Laurent polynomial.

## Theorem (Negut)

*The following family of Catalanimals lie in the shuffle algebra:*

$$\sigma_{\Gamma}^n\left(\frac{z^{\lambda}}{\prod_{i=1}^{n-1}(1 - qtz_i/z_{i+1})}\right) = H(R_+, R_+, R'_+, \lambda) \in \mathcal{S}_{\Gamma},$$

where  $R'_+ = \{\alpha_{ij} \in R_+ \mid i + 1 < j\}$ .

## The wheel condition

- A symmetric Laurent polynomial  $g(\mathbf{z})$  satisfies the *wheel condition* if it vanishes whenever any three of the variables  $z_i, z_j, z_k$  are in the ratio  $(z_i : z_j : z_k) = (1 : q : qt) = (1 : t : qt)$ .
- Let  $\mathcal{S}_{\check{\Gamma}} \cong \mathcal{S}_{\Gamma}$  for  
$$\check{\Gamma}(z_i, z_j) = (1 - z_i/z_j)(1 - qz_j/z_i)(1 - tz_j/z_i)(1 - qtz_i/z_j).$$

### Theorem (Negut)

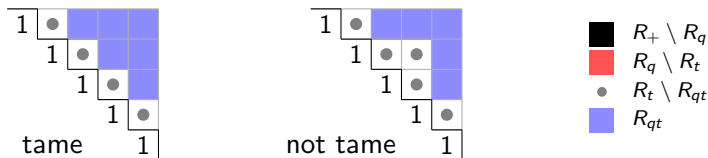
A symmetric Laurent polynomial  $g(z_1, \dots, z_n)$  belongs to  $\mathcal{S}_{\check{\Gamma}}$  if and only if it satisfies the wheel condition and vanishes whenever  $z_i = z_j$  for  $i \neq j$ .

# The wheel condition and tame Catalananimals

A Catalananimal  $H(R_q, R_t, R_{qt}, \lambda)$  is *tame* if

$$R_q + R_t \subseteq R_{qt},$$

where  $R_q + R_t = \{\alpha + \beta \mid \alpha \in R_q, \beta \in R_t\}$ .



The Catalananimals  $H(R_+, R_+, R'_+, \lambda)$  and the LLT Catalananimals are tame.

Using Negut's theorem, we show: Tame Catalananimals belong to the shuffle algebra  $\mathcal{S}_\Gamma$ .