RAMIFICATION OF PRIMES: A PRESENTATION FOR MATH 8600: COMMUTATIVE ALGEBRA

GEORGE H. SEELINGER

1. Introduction

Let $K|\mathbb{Q}$ be a finite field extension with $[K : \mathbb{Q}] = n$. Then, we may consider the integral closure of $\mathbb{Z}$ in K, say $\mathcal{O}_K$. Thus, we have the following setup.

$$\begin{array}{ccc} K & \hookleftarrow & \mathcal{O}_K \\ | & & | \\ \mathbb{Q} & \hookleftarrow & \mathbb{Z} \end{array}$$

where $\mathcal{O}_K|\mathbb{Z}$ is an integral ring extension. Now, recall the following facts.

1.1. **Proposition.** *Given the setup above*

(a) $\mathcal{O}_K$ *is a Dedekind domain.*
(b) Given a prime $p \in \mathbb{Z}$*, the ideal* $(p) = p\mathcal{O}_K \trianglelefteq \mathcal{O}_K$ *has a unique decomposition*

$$(p) = \prod_{i=1}^{g} P_i^{e_i}$$

for prime ideals $P_i \trianglelefteq \mathcal{O}_K$ *and* $e_i \in \mathbb{N}$.
(c) $\mathcal{O}_K$ *is a finitely-generated, free* $\mathbb{Z}$*-module, say*

$$\mathcal{O}_K \cong \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n \text{ as a } \mathbb{Z}\text{-module.}$$

Thus, $\mathcal{O}_K/p\mathcal{O}_K$ *is a finitely-generated* $\mathbb{Z}/p\mathbb{Z}$*-module, that is*

$$\mathcal{O}_K/p\mathcal{O}_K \cong (\mathbb{Z}/p\mathbb{Z})\overline{\alpha_1} \oplus \cdots \oplus (\mathbb{Z}/p\mathbb{Z})\overline{\alpha_n}$$

Furthermore, by the Chinese Remainder Theorem,

$$\mathcal{O}_K/p\mathcal{O}_k \cong \mathcal{O}_K/P_1^{e_1} \times \cdots \times \mathcal{O}_K/P_g^{e_g}$$

so each $\mathcal{O}_K/P_i^{e_i}$ *is an* $\mathbb{F}_p$*-vector space, and in fact, an* $\mathbb{F}_p$*-algebra since* $p \in P_i^{e_i}$.

This leads us to the following definition:

Date: May 2018.

1.2. **Definition.** We say a prime $p \in \mathbb{Z}$ is *ramified* in $\mathcal{O}_K$ if

$$p\mathcal{O}_K = \prod_{i=1}^{g} P_i^{e_i}$$

has some $e_i > 1$ for prime ideals $P_i \trianglelefteq \mathcal{O}_K$. If every $e_i = 1$, then p is *unramified* in $\mathcal{O}_K$.

1.3. **Example.** Consider $2 \in \mathbb{Z}[i]$. Then, since

$$-i(1+i)(1+i) = -i(1+2i-1) = -i2i = 2,$$

we have that $(2) \subseteq (1+i)^2$. Furthermore, since $(1+i)$ is prime in $\mathbb{Z}[i]$ using norm arguments, and (2) has norm 4, it must be that $(2) = (1+i)^2$. Therefore, 2 ramifies in $\mathbb{Z}[i]$.

We wish to come up with some method to determine when a prime will ramify in $\mathcal{O}_K$. One such characterization uses the notion of the "discriminant."

1.4. **Definition.** Let V be an m-dimensional vector space over K. Then, given a symmetric bilinear form $b\colon V \times V \to K$ and $\{\omega_1, \ldots, \omega_m\}$ a basis of V, we define

$$\operatorname{disc}(b; \omega_1, \ldots, \omega_m) := \det(b(\omega_i, \omega_j))_{1 \le i,j \le m}$$

1.5. **Proposition.** *Given another K-basis of V as above, say $\{\omega_1', \ldots, \omega_m'\}$ such that*

$$M \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_m \end{pmatrix} = \begin{pmatrix} \omega_1' \\ \vdots \\ \omega_m' \end{pmatrix}$$

we get that

$$\operatorname{disc}(b; \omega_1', \ldots, \omega_m') = (\det M)^2 \operatorname{disc}(b; \omega_1, \ldots, \omega_m)$$

Proof. Consider that if

$$B = (b(\omega_i, \omega_j))_{1 \le i,j \le m}, \quad B' = (b(\omega_i', \omega_j'))_{1 \le i,j \le m}$$

then,

$$B_{i,j}' = b(\omega_i', \omega_j') = b\left(\sum_{k=1}^{n} m_{k,i}\omega_k, \sum_{\ell=1}^{n} m_{\ell,j}\omega_\ell\right) = \sum_{k=1}^{n}\sum_{\ell=1}^{n} m_{i,k} b(\omega_k, \omega_\ell) m_{j,\ell} = (MBM^t)_{i,j}$$

and so $B' = MBM^t$. Then the result is obtained by taking the determinant of both sides. □

1.6. **Definition.** Let K be a field and let A be a finite-dimensional K-algebra with basis $\{x_1, \ldots, x_n\}$. Then,

(a) The *trace* $\operatorname{Tr}_{A|K}(z) := \operatorname{tr} m_z$ where, if

$$zx_i = \sum_{j=1}^{n} a_{i,j} x_j, \quad a_{i,j} \in K$$

then $m_z = (a_{i,j})_{1 \leq i,j \leq n}$. Note that this is independent of choice of basis since a different choice will give a matrix m_z' that is conjugate to m_z, which will not change the trace.

(b) The *trace form* $T \colon A \times A \to K$ is given by

$$T(x, y) = \operatorname{Tr}_{A|K}(xy)$$

Since we are in a commutative ring, the form is symmetric. Since matrix trace is bilinear, then so is the trace form.

(c) The *discriminant* of A is

$$\operatorname{disc}(A) := \operatorname{disc}(T; x_1, \ldots, x_n)$$

1.7. **Remark.** Consier the case that $K|\mathbb{Q}$ is a finite separable field extension with $\mathcal{O}_K \subseteq K$ the integral closure of $\mathbb{Z}$ in K.

(a) Then, the discriminant is independent of choice of integral basis since, given another integral basis $\{x_1', \ldots, x_n'\}$, we have

$$\operatorname{disc}(T; x_1', \ldots, x_n') = (\det M)^2 \operatorname{disc}(T; x_1, \ldots, x_n)$$

However, M is an invertible matrix with entries in $\mathbb{Z}$, so it must be that $\det M = \pm 1 \implies (\det M)^2 = 1$.

(b) Note $\operatorname{disc}(K)$ is always an integer because $\operatorname{Tr}_{K|\mathbb{Q}}(\mathcal{O}_K) \subseteq \mathbb{Z}$.

1.8. **Example.** Consider the field extension $\mathbb{Q}(i)|\mathbb{Q}$. Then, if we take integral basis $\{1, i\}$, we get

$$m_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, m_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } m_{-1} = -m_1$$

Thus,

$$\operatorname{Tr}(1) = 2, \operatorname{Tr}(i) = 0, \operatorname{Tr}(-1) = -2$$

and so

$$\operatorname{disc}(\{1, i\}) = \det \begin{pmatrix} \operatorname{Tr}(1) & \operatorname{Tr}(i) \\ \operatorname{Tr}(i) & \operatorname{Tr}(-1) \end{pmatrix} = \det \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = -4$$

This paper seeks to prove the following useful characterization for when a prime p ramifies in $\mathcal{O}_K$.

1.9. **Theorem.** *A prime $p \in \mathbb{Z}$ ramifies in $\mathcal{O}_K$ if and only if $p \mid \operatorname{disc}(K)$.*

From this result, we also have the useful corollary

1.10. **Corollary.** *Only a finite number of primes $p \in \mathbb{Z}$ ramify in $\mathcal{O}_K$.*

Thus, from our running example, 2 is the only prime that ramifies in $\mathbb{Z}[i]$. In the next section, we will follow a synthesis of the programs by [Ash03, 4.2] and [Con] to prove this theorem.

2. Structure and trace of the quotient $\mathcal{O}_K/p\mathcal{O}_K$

Using our same setup, let $(p) = p\mathcal{O}_K = \prod_i P_i^{e_i}$ for prime ideals $P_i \trianglelefteq \mathcal{O}_K$ and $e_i \in \mathbb{N}$.

2.1. **Lemma.** *p ramifies if and only if the ring $\mathcal{O}_K/(p)$ has nonzero nilpotent elements.*

Proof.

- ($\Longrightarrow$). Let p ramify in $\mathcal{O}_K$. Then, $\mathcal{O}_K/p\mathcal{O}_K \cong \mathcal{O}_K/P_1^{e_1} \times \cdots \times \mathcal{O}_K/P_n^{e_n}$ by the Chinese Remainder Theorem, where at least one $e_i > 1$, let us say e_1. Then, the quotient ring $\mathcal{O}_K/P_1^{e_1}$ has a nonzero nilpotent element since, for $x \in P_1 \backslash P_1^{e_1}$, we get $(x+P_1^{e_1})^{e_1} = x^{e_1} + P_1^{e_1} = P_1^{e_1}$.
- ($\Longleftarrow$). If p does not ramify in $\mathcal{O}_K$, then $\mathcal{O}_K/p\mathcal{O}_K \cong \mathcal{O}_K/P_1 \times \cdots \times \mathcal{O}_K/P_n$, each of which is a field since each P_i is maximal in $\mathcal{O}_K$. Furthermore, each of these fields is finite by Proposition 1.1(c). Thus, $\mathcal{O}_K/p\mathcal{O}_K$ cannot have any nonzero nilpotent elements.

$\square$

We also have, as a corollary to the proof, that

2.2. **Corollary.** *If p is unramified in $\mathcal{O}_K$, then $\mathcal{O}_K/p\mathcal{O}_K$ is a product of finite fields.*

This is a useful fact since

2.3. **Lemma.** *A nilpotent element has zero trace.*

Proof. Let $x^n = 0$ for some $n \in \mathbb{N}$. Then, since $m_{x^k} = (m_x)^k$, it must be that $(m_x)^n = 0$, so m_x is a nilpotent matrix, which has trace 0 since its mimimal polynomial $\mu_{m_x}(t) \mid t^n$. Therefore,

$$\mathrm{Tr}_{K|\mathbb{Q}}(x) = \mathrm{tr}\, m_x = 0$$

$\square$

And so, we get

2.4. **Lemma.** *For prime $p \in \mathbb{Z}$, let $p\mathcal{O}_K = \prod_{i=1}^{g} P_i^{e_i}$. For any $e_i > 1$,* $\mathrm{disc}_{\mathbb{F}_p}(\mathcal{O}_K/P_i^{e_i}) = \overline{0}$.

Proof. From 1.1(c), we have that $\mathcal{O}_K/P_i^{e_i}$ is an $\mathbb{F}_p$-algebra. By the above, since at least one $e_i > 1$, p ramifies and so we know $\mathcal{O}_K/P_i^{e_i}$ has a nonzero nilpotent element, say x. Then, extend $\{x\}$ to a basis of $\mathcal{O}_K/P_i^{e_i}$ over $\mathbb{F}_p$, say $\{x, x_2, \ldots, x_k\}$. Each xx_i is nilpotent, so, for all i,

$$\mathrm{Tr}_{\mathcal{O}_K/P_i^{e_i}|\mathbb{F}_p}(xx_i) = \overline{0}$$

and so, since the trace form matrix will have a row of all zeros, it must have determinant equal to $\overline{0}$ and so the discriminant is 0. $\square$

2.5. **Lemma.** *Let p is in $\mathcal{O}_K$ be unramified, that is, $p\mathcal{O}_K = \prod_{i=1}^{g} P_i$. Then, the trace form of $\mathcal{O}_K/P_i$ over $\mathbb{F}_p$ is nondegenerate. Thus, given the field extension $\mathcal{O}_K/P_i|\mathbb{F}_p$, the discriminant*

$$\operatorname{disc}(\mathcal{O}_K/P_i) \neq \overline{0} \in \mathbb{F}_p$$

Proof. By the arguments above, we already know that $\mathcal{O}_K/P_i$ is a finite field, and since $\mathbb{F}_p$ is perfect, we have that $\mathcal{O}_K/P_i|\mathbb{F}_p$ is a separable field extension. Therefore, by Lemma 2.2.3 in class, it must be that the trace form is nondegenerate. Therefore, fixing an $\mathbb{F}_p$-basis of $\mathcal{O}_K/P_i$, $\{\omega_1, \ldots, \omega_k\}$ the matrix

$$(T(\omega_i, \omega_j))_{1\leq i,j\leq n} \text{ is invertible } \iff \det(T(\omega_i, \omega_j))_{1\leq i,j\leq n} \neq \overline{0}$$

Therefore, $\operatorname{disc}(\mathcal{O}_K/P) \neq \overline{0}$. □

3. Discriminant Behaves Well with Reduction mod p and Products

3.1. **Lemma.** *For an appropriate choice of bases,*

$$\operatorname{disc}(K) \mod p = \operatorname{disc}(\mathcal{O}_K/p\mathcal{O}_K)$$

Proof. Let $\{\alpha_1, \ldots, \alpha_n\}$ be an integral basis for $\mathcal{O}_K|\mathbb{Z}$. Then, for $x \in \mathcal{O}_K$, we have $a_{i,j} \in \mathbb{Z}$ such that

$$x\alpha_i = \sum_j a_{i,j}\alpha_j \implies x\alpha_i + p\mathcal{O}_K = \sum_j \overline{a_{i,j}}\alpha_j + p\mathcal{O}_K$$

where $\overline{a_{i,j}} = a_{i,j} \mod p$. Thus, m_x with the entries reduced mod p is equal to $m_{x+p\mathcal{O}_K}$. Thus,

$$\operatorname{Tr}_{\mathcal{O}_K/p\mathcal{O}_K|\mathbb{F}_p}(x+p\mathcal{O}_K) = \operatorname{tr}(m_{x+p\mathcal{O}_K}) = \operatorname{tr}(m_x) \mod p = \operatorname{Tr}_{K|\mathbb{Q}}(x) \mod p$$

giving us that

$$(\operatorname{Tr}_{K|\mathbb{Q}}(\alpha_i\alpha_j))_{1\leq i,j\leq n} \mod p = \operatorname{Tr}_{\mathcal{O}_K/(p)|\mathbb{Z}/p\mathbb{Z}}(\overline{\alpha_i}\overline{\alpha_j})$$

and so, taking determinants of both sides gives the desired result. □

3.2. **Lemma.** *Let F be a field with B_1, B_2 finitely-generated F-algebras. Then, up to appropriate choice of basis,*

$$\operatorname{disc}(B_1 \times B_2) = \operatorname{disc}(B_1)\operatorname{disc}(B_2)$$

Proof. Let

$$B_1 = \bigoplus_{i=1}^{m} Fe_i, \quad B_2 = \bigoplus_{j=1}^{n} Ff_j$$

Then, take the standard choice of F-basis of $B_1\times B_2$, $\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$. Since $e_if_j = 0$ in $B_1 \times B_2$, we get that

$$\operatorname{disc}(B_1 \times B_2) = \det\begin{pmatrix} \operatorname{Tr}_{B_1\times B_2|F}(e_ie_k) & 0 \\ 0 & \operatorname{Tr}_{B_1\times B_2|F}(f_jf_\ell) \end{pmatrix}$$

Also, for $x \in B_1$, since $xy = 0$ for all $y \in B_2$, we have

$$\mathrm{Tr}_{B_1 \times B_2 | F}(x) = \mathrm{Tr}_{B_1|F}(x)$$

and similarly for $y \in B_2$

$$\mathrm{Tr}_{B_1 \times B_2 | F}(y) = \mathrm{Tr}_{B_2|F}(y)$$

Thus,

$$\begin{pmatrix} \mathrm{Tr}_{B_1 \times B_2|F}(e_i e_k) & 0 \\ 0 & \mathrm{Tr}_{B_1 \times B_2|F}(f_j f_\ell) \end{pmatrix} = \begin{pmatrix} \mathrm{Tr}_{B_1|F}(e_i e_k) & 0 \\ 0 & \mathrm{Tr}_{B_2|F}(f_j f_\ell) \end{pmatrix}$$

and so, taking the determinant of both sides, we get the desired result. □

4. Proof of the Ramification Theorem

We now prove our theorem.

Proof of 1.9. We first observe that

$$\begin{aligned} p \mid \mathrm{disc}(K) &\iff \mathrm{disc}(K) \equiv 0 \mod p \\ &\iff \mathrm{disc}(\mathcal{O}_K/(p)) = \overline{0} && \text{by Lemma 3.1} \\ &\iff \prod \mathrm{disc}(\mathcal{O}_K/P_i^{e_i}) = \overline{0} && \text{by Lemma 3.2} \end{aligned}$$

Thus, if any $e_i > 1$, we get that $\mathcal{O}_K/P_i^{e_i}$ has a nonzero nilpotent element by 2.1, and so $\mathrm{disc}(\mathcal{O}_K/P_i^{e_i}) = \overline{0}$ by 2.4, thus giving $p \mid \mathrm{disc}_{\mathbb{Z}}(\mathcal{O}_K)$ by the equivalences above.

If all $e = 1$, then each $\mathcal{O}_K/P_i$ is a finite field, so $\mathrm{disc}(\mathcal{O}_K/P_i) \neq \overline{0}$ by 2.5. Therefore, it must be that $p \nmid \mathrm{disc}(K)$. □

5. Factorization in Quadratic Number Fields

In this section, we follow [Ash03] to determine some results about factorization of primes in quadratic number fields. First, recall the theorem

5.1. **Theorem** (Ram-Rel Identity)**.** *Let A be an integral domain with field of fractions K, $L|K$ a finite separable field extension of degree n, and B the integral closure of A in L. Given a prime ideal $P \trianglelefteq A$, if*

$$PB = \prod_{i=1}^{g} P_i^{e_i} \quad f_i = [B/P_i : A/P]$$

then

$$\sum_{i=1}^{g} e_i f_i = [B/PB : A/P] = n$$

Thus, for $m \in \mathbb{Z} \setminus \{0, 1\}$, a squarefree integer, $\mathbb{Q}(\sqrt{m})|\mathbb{Q}$ has degree 2. Thus, for a prime $p \in \mathbb{Z}$, there are only three possible situations.

(a) $g = 2, e_1 = e_2 = f_1 = f_2 = 1$, that is,

$$(p) = P_1 P_2$$

In this situation, we say that p *splits* in $\mathcal{O}_K$.

(b) $g = 1, e_1 = 1, f_1 = 2$, that is, (p) is a prime ideal of $\mathcal{O}_K$. In this situations, we say that (p) is *inert*.

(c) $g = 1, e_1 = 2, f_1 = 1$, that is,

$$(p) = P_1^2$$

so p ramifies.

Furthermore, we will use the following result about the discriminant of $\mathbb{Q}(\sqrt{m})$.

5.2. **Proposition.** *The discriminant of* $\mathbb{Q}(\sqrt{m})$ *is* m *if* $m \equiv 1 \mod 4$ *and it is* $4m$ *if* $m \equiv 2, 3 \mod 4$. *In particular, the discriminant is always* 0 *or* $1 \mod 4$.

Proof. If $m \not\equiv 1 \mod 4$, $\{1, \sqrt{m}\}$ is an integral basis of $\mathbb{Q}(\sqrt{m})$. Then,

$$\mathrm{Tr}(a+b\sqrt{m}) = \mathrm{tr}\begin{pmatrix} a & b \\ bm & a \end{pmatrix} = 2a \Longrightarrow \mathrm{disc}(\mathbb{Q}(\sqrt{m})) = \det\begin{pmatrix} 2 & 0 \\ 0 & 2m \end{pmatrix} = 4m$$

If $m \equiv 1 \mod 4$, then $\{1, \frac{1+\sqrt{m}}{2}\}$ forms an integral basis and

$$\left(\frac{1+\sqrt{m}}{2}\right)^2 = \frac{m-1}{4} + \frac{1+\sqrt{m}}{2}$$

So, $\mathrm{Tr}(1) = 2$ and

$$\mathrm{Tr}\left(\frac{1+\sqrt{m}}{2}\right) = \mathrm{tr}\begin{pmatrix} 0 & 1 \\ \frac{m-1}{4} & 1 \end{pmatrix} = 1,$$

$$\mathrm{Tr}\left(\frac{m-1}{4} + \frac{1+\sqrt{m}}{2}\right) = \mathrm{tr}\begin{pmatrix} \frac{m-1}{4} & 1 \\ \frac{m-1}{4} & \frac{m+3}{4} \end{pmatrix} = \frac{m+1}{2}$$

Thus

$$\mathrm{disc}(\mathbb{Q}(\sqrt{m})) = \det\begin{pmatrix} 2 & 1 \\ 1 & \frac{1+m}{2} \end{pmatrix} = m$$

□

We then have the following result.

5.3. **Theorem.** *Let prime* $p \neq 2$. *Then,*

(a) (p) *ramifies as* $(p, \sqrt{m})^2$ *in* $\mathbb{Q}(\sqrt{m})$ *if and only if* $m \equiv 0 \mod p$.

(b) (p) *splits as* $(p) = (p, a + \sqrt{m})(p, a - \sqrt{m})$ *in* $\mathbb{Q}(\sqrt{m})$ *if and only if* $m \equiv a^2 \mod p$ *for some* $a \not\equiv 0 \mod p$.

(c) (p) *is inert in* $\mathbb{Q}(\sqrt{m})$ *if and only if* $m \not\equiv a^2 \mod p$ *for all* a.

If $p = 2$ *and* m *is odd, then*

(a) (2) *ramifies in* $\mathbb{Q}(\sqrt{m})$ *if and only if* $m \equiv 3 \mod 4$.

(b) (2) *splits as* $\left(2, \frac{1+\sqrt{m}}{2}\right)\left(2, \frac{1-\sqrt{m}}{2}\right)$ *in* $\mathbb{Q}(\sqrt{m})$ *if and only if* $m \equiv 1 \mod 8$.

(c) (2) *is inert in* $\mathbb{Q}(\sqrt{m})$ *if and only if* $m \equiv 5 \mod 8$.

Proof. We break down the various situations. Throughout, let $D = \text{disc}(\mathbb{Q}(\sqrt{m}))$.

- Assume p is an odd prime with p not dividing m. p does not divide the discriminant, so (p) cannot ramify.
  - If $m \equiv a^2 \mod p$, $a \not\equiv 0 \mod p$, then $(p) = (p, a+\sqrt{m})(p, a-\sqrt{m})$ because

$$(p, a+\sqrt{m})(p, a-\sqrt{m}) = (p^2, pa + p\sqrt{m}, pa - p\sqrt{m}, \underbrace{a^2 - m}_{\equiv 0 \mod p}) \subseteq (p)$$

  and since

$$p(a+\sqrt{m}+a-\sqrt{m}) = 2ap \in (p, a+\sqrt{m})(p, a-\sqrt{m})$$

  but $a \not\equiv 0 \mod p$, so $\gcd(2ap, p^2) = p$, and thus $p \in (p, a+\sqrt{m})(p, a-\sqrt{m})$.
  - If $m \not\equiv a^2 \mod p$, then $x^2 - m$ is irreducible $\mod p$. Assume $(p) = Q_1Q_2$. Each Q_i must have norm p, thus giving $\mathcal{O}_K/Q_i \cong \mathbb{F}_p$. However, $\sqrt{m} \in \mathcal{O}_K \implies m$ has a square root in $\mathbb{F}_p$, a contradiction. Thus, (p) is inert.
- Let p divide m. Then, p divides the discriminant and so (p) ramifies. In fact,

$$(p, \sqrt{m})^2 = (p^2, p\sqrt{m}, m) \subseteq (p)$$

  However, since m is squarefree, $p^2 \nmid m$, so $\gcd(p^2, m) = p$, so $p \in (p, \sqrt{m})^2$.
- Let $p = 2$ and m be odd.
  - If $m \equiv 3 \mod 4 \implies D = 4m$, then 2 divides the discriminant, so (2) ramifies. We claim $(2) = (2, 1+\sqrt{m})^2$. First, we check

$$(2, 1+\sqrt{m})^2 = (4, 2(1+\sqrt{m}), \underbrace{1+2\sqrt{m}+m}_{\equiv 0 \mod 2}) \subseteq (2)$$

  Furthermore,

$$1 + 2\sqrt{m} + m - 2(1+\sqrt{m}) = m - 1 \equiv 2 \mod 4$$

  so there is some $x \in \mathbb{Z}$ such that

$$m - 1 + 4x = 2$$

  thus giving us equality of ideals.
  - If $m \equiv 1 \mod 8$, then $m \equiv 1 \mod 4$, so we get an integral basis $\{1, \frac{1+\sqrt{m}}{2}\}$ and the discriminant is $D = m$. Therefore, $2 \nmid D$, so (2) does not ramify. We then compute,

$$(2, \frac{1+\sqrt{m}}{2})(2, \frac{1-\sqrt{m}}{2}) = (4, 1-\sqrt{m}, 1+\sqrt{m}, \underbrace{\frac{1-m}{4}}_{\text{Even}}) \subseteq (2)$$

However, we also have

$$1-\sqrt{m}+1+\sqrt{m}=2\in(2,\frac{1+\sqrt{m}}{2})(2,\frac{1-\sqrt{m}}{2})$$

giving us the desired ideal equality.

– If $m \equiv 5 \mod 8$, then $m \equiv 1 \mod 4$, so $D = m$, meaning 2 does not ramify. Consider

$$f(x)=x^2-x+\frac{1-m}{4}\in(\mathcal{O}_K/P)[x]$$

where $(2)\subseteq P$ a prime ideal in $\mathcal{O}_K$. The roots of f are $\frac{1\pm\sqrt{m}}{2}$, so f has a root in $\mathcal{O}_K$ and hence in $\mathcal{O}_K/P$. However, since $\frac{1-m}{4}\equiv 1 \mod 2$, f has no root in $\mathbb{F}_2$. Therefore, $\mathcal{O}_K/P$ and $\mathbb{F}_2$ cannot be isomorphic. If $(2)=P_1P_2$ in $\mathcal{O}_K$, then the norm of (2) is 4 and so P_1, P_2 each have norm 2. Therefore, $\mathcal{O}_K/P_i\cong\mathbb{F}_2$, which is a contradiction. Thus, (2) must remain prime.

□

References

[Ash03] R. B. Ash, *A Course In Algebraic Number Theory*, 2003. https://faculty.math.illinois.edu/~r-ash/ANT.html.

[Con] K. Conrad, *Discriminants and Ramified Primes*. http://www.math.uconn.edu/~kconrad/blurbs/gradnumthy/disc.pdf.

[MC16] S. Mack-Crane, *Prime Splitting in Quadratic Extensions I: One Prime, Many Fields* (2016). https://algebrateahousejmath.wordpress.com/2016/11/23/prime-splitting-in-quadratic-extensions-i-one-prime-many-fields/.