Catalanimals, shuffle theorems, and Macdonald polynomials

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SMRI: Modern Perspectives in Representation Theory

22 May 2025

Overview

Algebraic quantity = Combinatorial generating function

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Representation theory

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\begin{tikzcd}
\text{Closed form expression} \arrow[d] & = & \text{Infinite symmetric series} \arrow[d] \\
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Outline

1. **Background on symmetric functions and Macdonald polynomials**
2. Shuffle theorems, combinatorics, and LLT polynomials
3. A new formula for Macdonald polynomials

Symmetric polynomials and functions

- Symmetric polynomials $\mathbb{k}[z_1, \ldots, z_n]^{S_n}$
- Generators

$$e_r(z_1, \ldots, z_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} z_{i_1} \cdots z_{i_r}$$

$$\begin{aligned} e_1(z_1, z_2, z_3) &= z_1 + z_2 + z_3 \\ e_2(z_1, z_2, z_3) &= z_1 z_2 + z_1 z_3 + z_2 z_3 \\ e_3(z_1, z_2, z_3) &= z_1 z_2 z_3 \end{aligned}$$

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- Integer partitions of d.

Partitions

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

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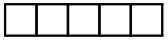
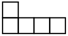
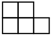
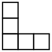
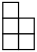
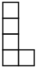
$5 \to$ 

$4 + 1 \to$ 

$3 + 2 \to$ 

$3 + 1 + 1 \to$ 

$2 + 2 + 1 \to$ 

$2 + 1 + 1 + 1 \to$ 

$1 + 1 + 1 + 1 + 1 \to$ 

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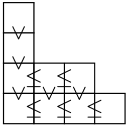

For $\lambda = (2, 1)$,

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Hidden Guide: Schur Positivity

"Naturally occurring" symmetric functions which are non-negative (coefficients in $\mathbb{N}$) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

A Graded Example

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

$S_3 \curvearrowright \mathsf{span}\{\Delta\}$ via $\sigma.\Delta = \mathrm{sgn}(\sigma)\Delta$.

A Graded Example

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

$S_3 \curvearrowright \text{span}\{\Delta\}$ via $\sigma.\Delta = \mathrm{sgn}(\sigma)\Delta$.
Consider $S_3 \curvearrowright M =$ on all partial derivatives of Δ.

$$\begin{array}{c} \text{span}\{\Delta\} \\ \big\downarrow \partial_{x_i} \\ \text{span}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\} \\ \big\downarrow \partial_{x_i} \\ \text{span}\{x_3 - x_1, x_2 - x_3\} \\ \big\downarrow \partial_{x_i} \\ \text{span}\{1\} \end{array}$$

A Graded Example

$$\begin{array}{cr}
\mathrm{span}\{\Delta\} & \cong V_{\ydiagram{1,1,1}} \\
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- $\operatorname{Frob}(M) = s_{\ydiagram{1,1,1}} + 2s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$
- Problem: what about grading?

Getting More Information

$$\begin{array}{ccc}
\operatorname{span}\{\Delta\} & \cong V_{\ydiagram{1,1,1}} & \deg = 3 \\
\big\downarrow{\scriptstyle \partial_{x_i}} & & \\
\operatorname{span}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\} & \cong V_{\ydiagram{1,2}} & \deg = 2 \\
\big\downarrow{\scriptstyle \partial_{x_i}} & & \\
\operatorname{span}\{x_3 - x_1, x_2 - x_3\} & \cong V_{\ydiagram{1,2}} & \deg = 1 \\
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- $(V_\lambda \text{ in degree } \color{blue}{d}) \mapsto q^{\color{blue}{d}} s_\lambda$
- $\mathrm{Frob}(M) = q^{\color{blue}{3}} s_{\ydiagram{1,1,1}} + q^{\color{blue}{2}} s_{\ydiagram{1,2}} + q^{\color{blue}{1}} s_{\ydiagram{1,2}} + q^{\color{blue}{0}} s_{\ydiagram{3}}$.
- Hall-Littlewood polynomial $H_{\ydiagram{1,1,1}}(X; q)$.
- Remark: $M \cong \mathbb{Z}[x_1, x_2, x_3]/\mathrm{Sym}^+ \cong H^*(Fl_3)$ as **graded** S_3-representations.

A Problem

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- $\tilde{H}_\lambda(X;1,1) = s_1^{|\lambda|}$.
- Does there exist a family of S_n-regular representations whose bigraded Frobenius characteristics equal $\tilde{H}_\lambda(X;q,t)$?

Garsia-Haiman modules

- $S_n \curvearrowright \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

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$M_{2,1} =$

$$\underbrace{\mathrm{span}\{\Delta_{2,1}\}}_{\deg=(1,1)} \oplus \underbrace{\mathrm{span}\{y_3 - y_1, y_1 - y_2\}}_{\deg=(0,1)} \oplus \underbrace{\mathrm{span}\{x_3 - x_1, x_1 - x_2\}}_{\deg=(1,0)} \oplus \underbrace{\mathrm{span}\{1\}}_{\deg=(0,0)}$$

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Irreducible S_n-representation V_λ with bidegree $({\color{blue}a}, {\color{orange}b}) \mapsto q^{\color{blue}a} t^{\color{orange}b} s_\lambda$

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$$\tilde{H}_{\ydiagram{1,2}} = q^1 t^1 s_{\ydiagram{1,1,1}} + t^1 s_{\ydiagram{1,2}} + q^1 s_{\ydiagram{1,2}} + s_{\ydiagram{3}}$$

Garsia-Haiman modules

Theorem (Haiman, 2001)

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$\tilde{H}_\lambda(X;q,t) = \sum_\mu \tilde{K}_{\lambda\mu}(q,t)s_\mu$ *satisfies* $\tilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$.

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- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
s_λ	Irreducible V_λ	SSYT(λ)
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	Later

Garsia-Haiman modules

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \mathrm{span}\{f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \mid \left(\sum_{j=1}^{n} \partial^r_{x_j} \partial^s_{y_j}\right) f = 0, \forall r+s > 0\}$$

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Question

What symmetric function is the bigraded Frobenius characteristic of DH_n?

∇e_n

Frobenius characteristic of DH_3

∇e_n

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$$= \frac{t^3\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2t+qt^2+qt)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{q^3\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

∇e_n

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Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q+t+1)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

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Operator ∇

$$\nabla \tilde{H}_\lambda(X;q,t) = q^{n(\lambda)} t^{n(\lambda^*)} \tilde{H}_\lambda(X;q,t)\,,$$

where $n(\lambda) = \sum_i (i-1)\lambda_i$ and λ^* is the transpose partition to λ.

∇e_n

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Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
s_λ	Irreducible V_λ	$\mathrm{SSYT}(\lambda)$
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	Later
∇e_n	DH_n	Now: Shuffle theorem

Outline

1. Background on symmetric functions and Macdonald polynomials
2. **Shuffle theorems, combinatorics, and LLT polynomials**
3. A new formula for Macdonald polynomials

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k = \sum_{\lambda} (q, t \ \textit{monomial})(\textit{LLT polynomial})$$

- Summation over all k-by-k Dyck paths.

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} (\textit{LLT polynomial})$$

- Summation over all k-by-k Dyck paths.
- $\mathrm{area}(\lambda)$ and $\mathrm{dinv}(\lambda)$ statistics of Dyck paths.

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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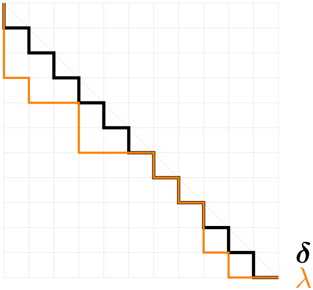
- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



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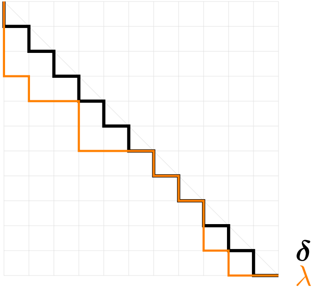

- $\text{area}(\lambda) =$ number of squares above λ but below the path δ of alternating S-E steps.

Dyck paths

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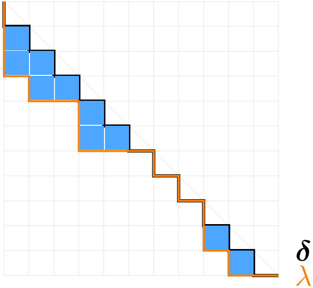
A Dyck path λ is a south-east lattice path lying below the line segment from $(0,k)$ to $(k,0)$.



- $\text{area}(\lambda)$ = number of squares above λ but below the path δ of alternating S-E steps.
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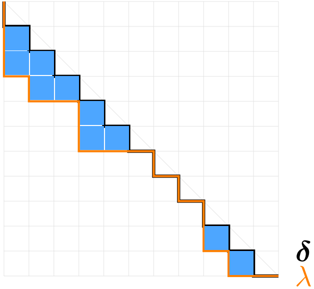

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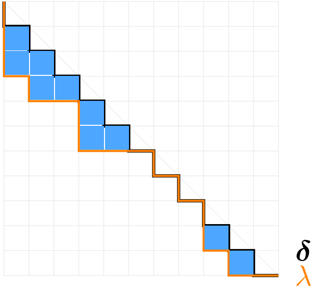
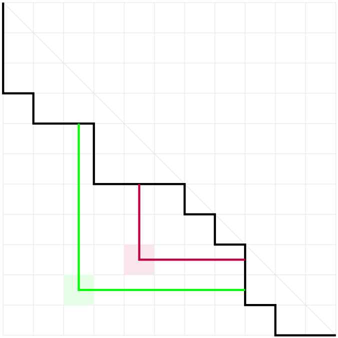


- area(λ) = number of squares above λ but below the path δ of alternating S-E steps.
- E.g., above area$(\lambda) = 10$.
- Catalan-number many Dyck paths for fixed k. (1,2,5,14,42,...)

dinv

dinv(λ) $=\#$ of balanced hooks in diagram below λ.



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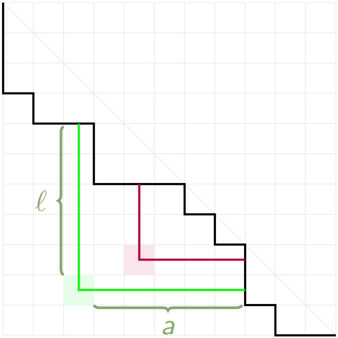
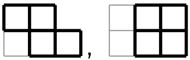


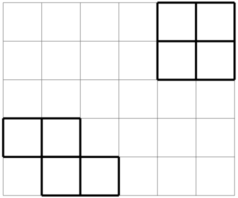
Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1-\epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

Key Object: LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)

$\boldsymbol{\nu} = \Bigg($  $\Bigg)$



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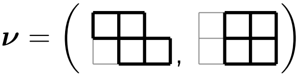

-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

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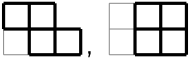
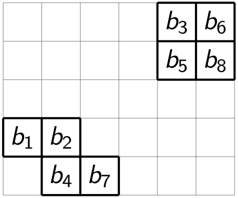
$\boldsymbol{\nu} = \Bigg($  $\Bigg)$



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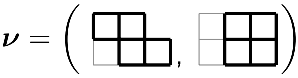

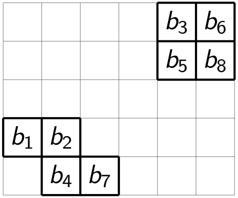

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

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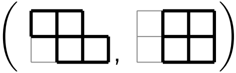
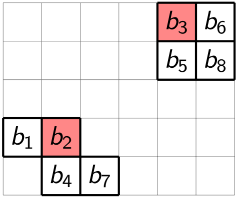
$\boldsymbol{\nu} =$ 



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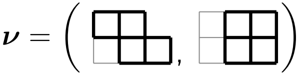

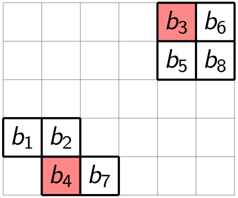

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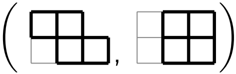
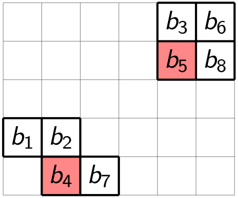
$\boldsymbol{\nu} =$ 



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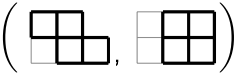
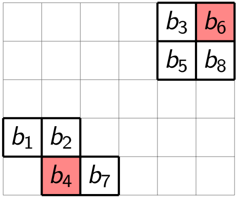
$\boldsymbol{\nu} =$ 



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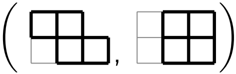
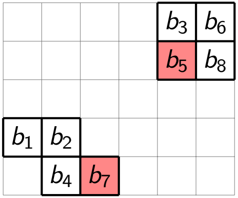
$\boldsymbol{\nu} =$ 



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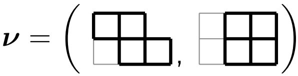

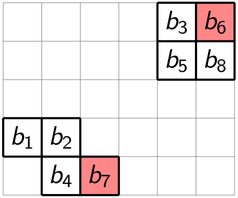

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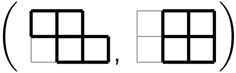
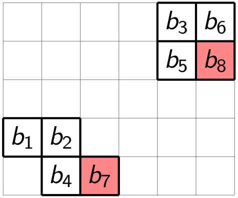
$\boldsymbol{\nu} =$ 



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LLT Polynomials

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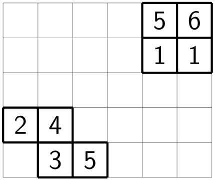
$T =$ 

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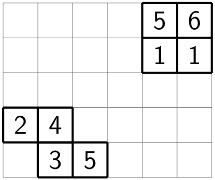

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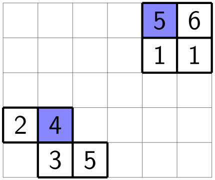

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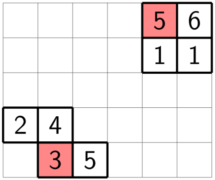
$T =$ 

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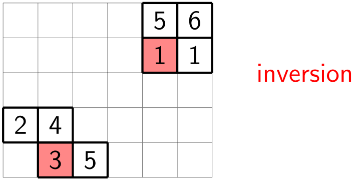

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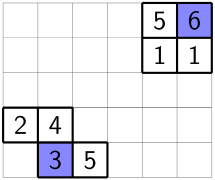

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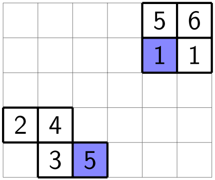

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- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
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The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{z}; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{z}^T,$$

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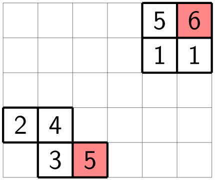

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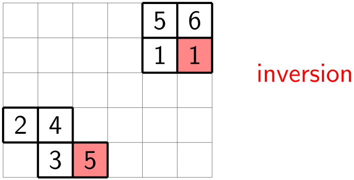

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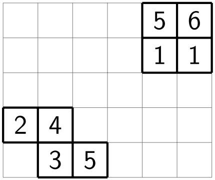
$T \quad =$ 

$\mathrm{inv}(T) = 4, \quad \boldsymbol{z}^T = z_1^2 z_2 z_3 z_4 z_5^2 z_6$

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Example ∇e_3

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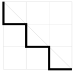		
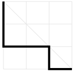		
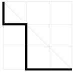		
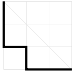		
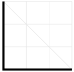		

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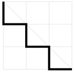
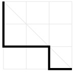
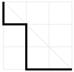
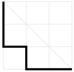
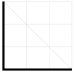
	q^3	
	$q^2 t$	
	qt	
	qt^2	
	t^3	

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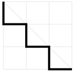
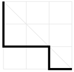
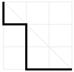
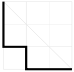
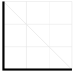
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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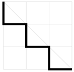
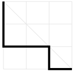
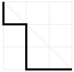
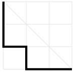
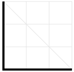
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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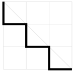
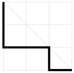
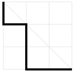
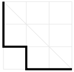
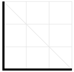
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

- Entire quantity is q, t-symmetric
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What generalizes ∇e_k ?

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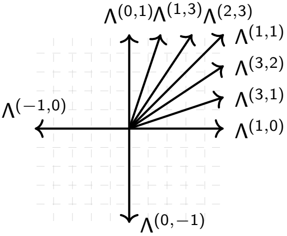

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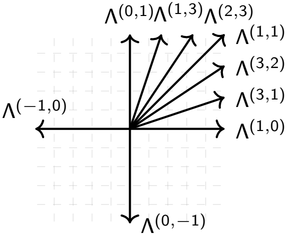

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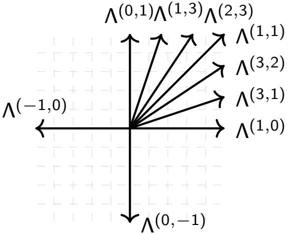

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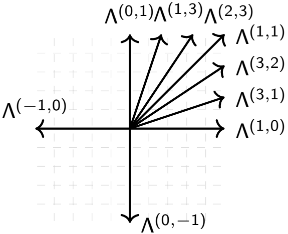

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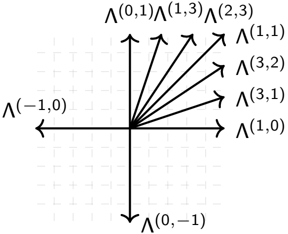

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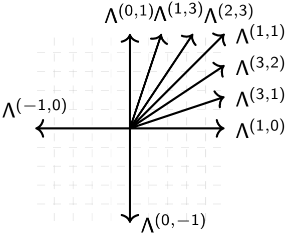

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- $e_k^{(1,1)} \cdot 1 = \nabla e_k^{(0,1)} \nabla^{-1} \cdot 1 = \nabla e_k$.
- Can be difficult to work with in general. Can we make it more explicit?

Root ideals

$R_+ = \left\{\alpha_{ij} \mid 1 \leq i < j \leq n\right\}$ denotes the set of positive roots for GL_n, where $\alpha_{ij} = \epsilon_i - \epsilon_j$.

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

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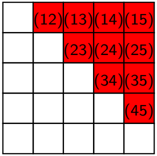
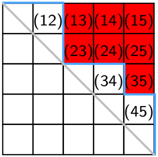


A root ideal $\Psi \subseteq R_+$ is an upper order ideal of positive roots.



Ψ = Roots above Dyck path

Weyl symmetrization

Define the *Weyl symmetrization operator*
$\boldsymbol{\sigma} \colon \mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \to \mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]^{S_n}$ by

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- For λ a partition, $\chi_\lambda(\boldsymbol{z}) = s_\lambda(\boldsymbol{z})$.

Weyl symmetrization

Define the *Weyl symmetrization operator*
$\boldsymbol{\sigma} : \mathbb{Q}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \to \mathbb{Q}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]^{S_n}$ by

$$f(\boldsymbol{z}) \mapsto \sum_{w \in S_n} w \left(\frac{f(\boldsymbol{z})}{\prod_{i<j}(1 - z_j/z_i)} \right) .$$

$\boldsymbol{\sigma}(z_1^{\gamma_1} \cdots z_n^{\gamma_n}) = \pm$ irreducible GL_n character or 0.

Example

$\boldsymbol{\sigma}(\boldsymbol{z}^{111} + \boldsymbol{z}^{201} + \boldsymbol{z}^{210} + \boldsymbol{z}^{3\text{-}11}) = \chi_{111} + 0 + \chi_{210} - \chi_{300}$

- For λ a partition, $\chi_\lambda(\boldsymbol{z}) = s_\lambda(\boldsymbol{z})$.
- $\mathrm{pol}_X\, \chi_\lambda(\boldsymbol{z}) = s_\lambda$ if $\lambda_l \geq 0$, otherwise 0.

Catalanimals

Definition

The *Catalanimal* indexed by $R_q, R_t, R_{qt} \subseteq R_+$ and $\lambda \in \mathbb{Z}^n$ is

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where $\boldsymbol{z}^{\alpha_{ij}} = z_i/z_j$ and $(1 - tz_i/z_j)^{-1} = 1 + tz_i/z_j + t^2 z_i^2/z_j^2 + \cdots$.

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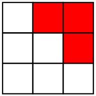
With $n = 3$, $R_+ =$ 

$H(R_+, R_+, \{\alpha_{13}\}, (111)) =$

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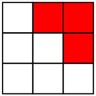
With $n = 3$, $R_+ =$ 

$$H(R_+, R_+, \{\alpha_{13}\}, (111)) = \mathrm{pol}_X\, \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^{111}(1 - qtz_1/z_3)}{\prod_{1 \le i < j \le 3}(1 - qz_i/z_j)(1 - tz_i/z_j)} \right)$$

$$= s_{111} + (q + t + q^2 + qt + t^2)s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3)s_3$$

$$= \omega \nabla e_3.$$

Why?

Let $R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq l\}$ and $R_+^0 = \{\alpha_{ij} \in R_+ \mid i + 1 < j\}$.

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Proposition

For $(m, n) \in \mathbb{Z}_+^2$ *coprime,*

$$e_k^{(m,n)} \cdot 1 = H(R_+, R_+, R_+^0, \mathbf{b})$$

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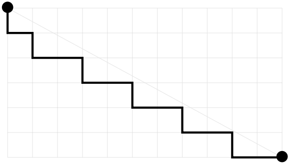
for $\mathbf{b} = (b_0, \ldots, b_{km-1})$ satisfying $b_i =$ the number of south steps on vertical line $x = i$ of highest lattice path under line $y + \frac{n}{m}x = n$.

$\delta =$ highest Dyck path.



δ

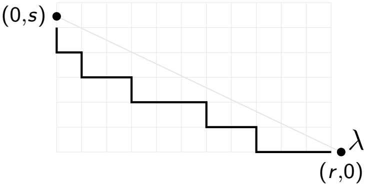
$$\mathbf{b} = (1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0)$$

Results

Manipulating Catalanimal $\Longrightarrow$ a proof of the Rational Shuffle Theorem + a generalization.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2023)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $\mathbf{b} = (b_1, \ldots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.



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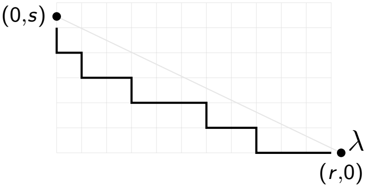

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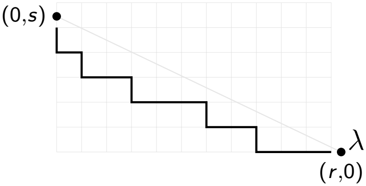
$$H(R_+, R_+, R_+^0, \mathbf{b}) =$$



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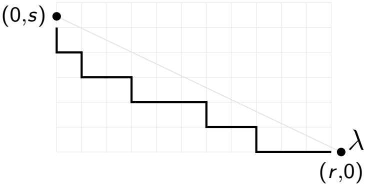
$$H(R_+, R_+, R_+^0, \mathbf{b}) = \sum_{\lambda} \qquad \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line $y + px = s$,



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$$H(R_+, R_+, R_+^0, \mathbf{b}) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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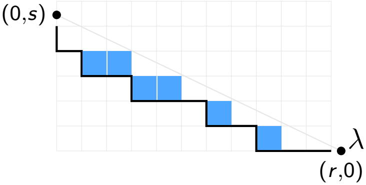

area(λ) as before

$\mathrm{dinv}_p(\lambda) = \#p$-balanced hooks

$\frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

A Question

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Special case: $\mathcal{G}_{\boldsymbol{\nu}}^{(1,1)} \cdot 1 = \nabla \mathcal{G}_{\boldsymbol{\nu}}(X; q)$.

LLT Catalanimals

For a tuple of skew shapes $\boldsymbol{\nu}$, the *LLT Catalanimal* $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ is determined by

- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$,

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- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$,
- $R_+ \setminus R_q =$ pairs of boxes in the same diagonal in the same shape,
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  $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.
  Listing this filling in reading order gives λ.

LLT Catalanimals

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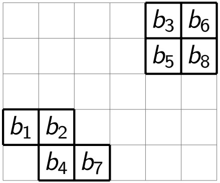
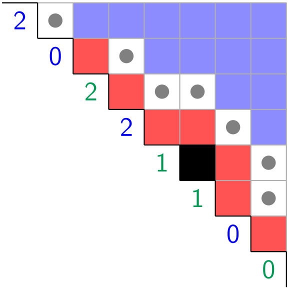


$\boldsymbol{\nu}$



LLT Catalanimals

- (black) $R_+ \setminus R_q$ = pairs of boxes in the same diagonal in the same shape,
- (red) $R_q \setminus R_t$ = the attacking pairs,
- • $R_t \setminus R_{qt}$ = pairs going between adjacent diagonals,
- (blue) R_{qt} = all other pairs,

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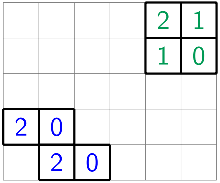
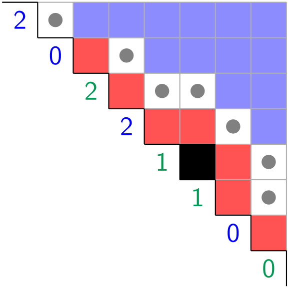


λ, as a filling of $\boldsymbol{\nu}$



LLT Catalanimals

Theorem (Blasiak-Haiman-Morse-Pun-S., 2024)

Let $\boldsymbol{\nu}$ be a tuple of skew shapes and let $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ be the associated LLT Catalanimal. Then

$$\begin{aligned}\nabla \mathcal{G}_{\boldsymbol{\nu}}(X;q) &= c_{\boldsymbol{\nu}}\, \omega H_{\boldsymbol{\nu}} \\ &= c_{\boldsymbol{\nu}}\, \omega \boldsymbol{\sigma} \left(\frac{\boldsymbol{z}^{\lambda} \prod_{\alpha \in R_{qt}} \left(1 - qt\, \boldsymbol{z}^{\alpha}\right)}{\prod_{\alpha \in R_q} \left(1 - q\, \boldsymbol{z}^{\alpha}\right) \prod_{\alpha \in R_t} \left(1 - t\, \boldsymbol{z}^{\alpha}\right)} \right)\end{aligned}$$

for some $c_{\boldsymbol{\nu}} \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$.

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- Remember $\nabla \tilde{H}_\mu = q^{n(\mu)} t^{n(\mu^*)} \tilde{H}_\mu$.

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Outline

1. Background on symmetric functions and Macdonald polynomials
2. Shuffle theorems, combinatorics, and LLT polynomials
3. **A new formula for Macdonald polynomials**

Haglund-Haiman-Loehr formula example

$\tilde{H}_\mu(X;q,t) = \sum_D \left(\prod_{u\in D} q^{-\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1}\right) \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}(X;q)$

Haglund-Haiman-Loehr formula example

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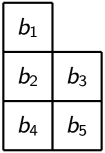
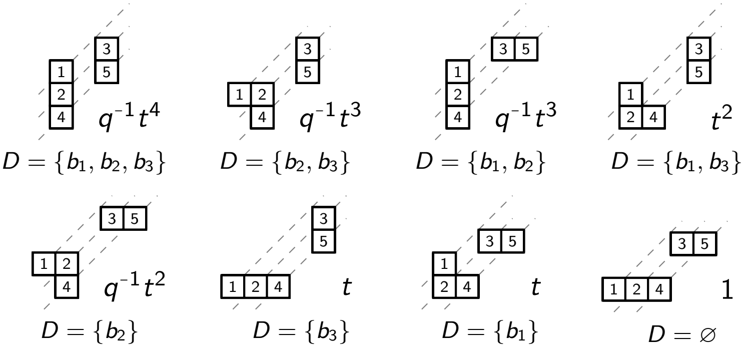


μ



Putting it all together

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}$ and apply $\omega\nabla$.

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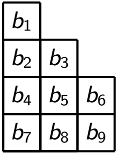
- By construction, all the LLT Catalanimals $H_{\boldsymbol{\nu}(\mu,D)}$ appearing on the RHS will have the same root ideal data (R_q, R_t, R_{qt}).
- Collect terms to get $\prod_{(b_i,b_j)\in V(\mu)}(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j)$ factor for $V(\mu)$ the set of vertical dominoes (b_i, b_j) in μ.

$$\tilde{H}_\mu = \omega\, \mathrm{pol}_X\, \boldsymbol{\sigma}\left(z_1 \cdots z_n \frac{\prod\limits_{\alpha_{ij}\in V(\mu)} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod\limits_{\alpha\in \widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+}\left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha\in R_\mu}\left(1 - t\boldsymbol{z}^\alpha\right)} \right).$$

The root ideal R_μ



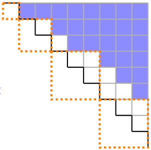
row reading order

$b_1 \prec b_2 \prec \cdots \prec b_n$

$$R_\mu := \{\alpha_{ij} \in R_+ \mid \mathrm{south}(b_i) \preceq b_j\},$$
$$\widehat{R}_\mu := \{\alpha_{ij} \in R_+ \mid \mathrm{south}(b_i) \prec b_j\},$$
$$R_\mu \setminus \widehat{R}_\mu \leftrightarrow V(\mu) = \text{vertical dominoes in } \mu$$

Example:

$R_{3321} =$



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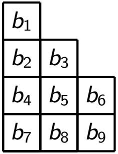

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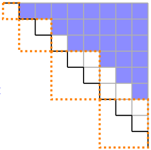
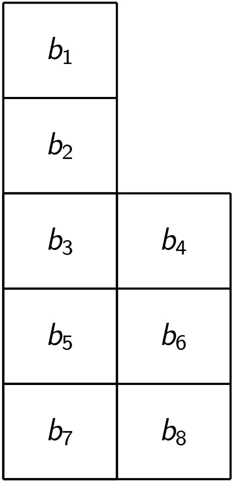
$R_{3321} =$ 

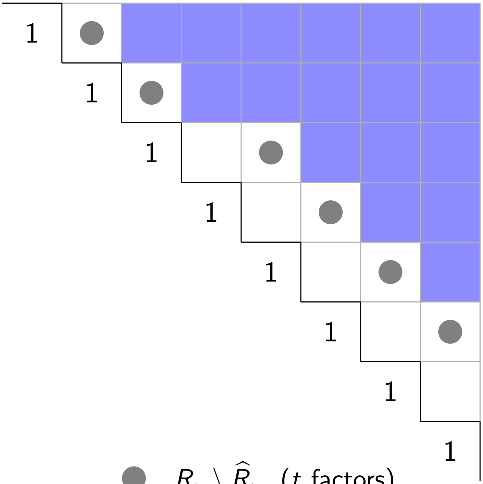
Remark

$$\tilde{H}_\mu(X;0,t) = \omega \, \mathrm{pol}_X \, \boldsymbol{\sigma}\left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu}(1 - t\boldsymbol{z}^\alpha)}\right)$$

Example



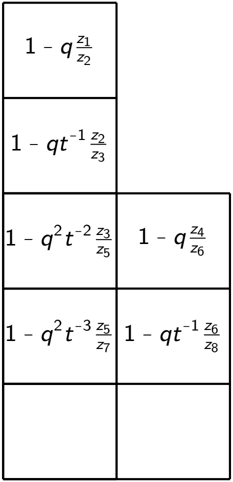
partition $\mu = 22211$



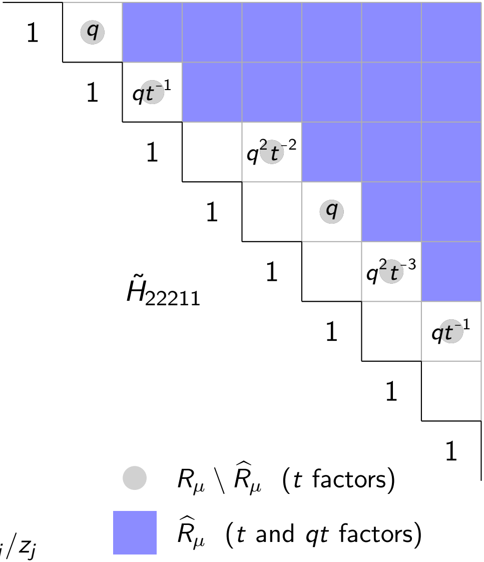
$R_\mu \setminus \widehat{R}_\mu$ (t factors)

$\widehat{R}_\mu$ (t and qt factors)

Example



numerator factors $1 - q^{\mathrm{arm}+1}t^{-\mathrm{leg}}z_i/z_j$



A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

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$$\tilde{H}_{\mu}^{(s)} = \omega \, \mathrm{pol}_X \, \boldsymbol{\sigma} \left((z_1 \cdots z_n)^s \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod_{\alpha \in \widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^{\alpha}\right)}{\prod_{\alpha \in R_+} \left(1 - q\boldsymbol{z}^{\alpha}\right) \prod_{\alpha \in R_\mu} \left(1 - t\boldsymbol{z}^{\alpha}\right)} \right)$$

Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition μ and positive integer s, the symmetric function $\tilde{H}_{\mu}^{(s)}$ is Schur positive. That is, the coefficients in

$$\tilde{H}_{\mu}^{(s)} = \sum_{\nu} K_{\nu,\mu}^{(s)}(q,t)\, s_\nu$$

satisfy $K_{\nu,\mu}^{(s)}(q,t) \in \mathbb{N}[q,t]$.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
s_λ	Irreducible V_λ	SSYT(λ)
$\tilde{H}_\lambda(X;q,t)$	Garsia-Haiman M_λ	HHL
∇e_n	DH_n	Shuffle theorem
$H(R_+, R_+, R_+^0, \mathbf{b})$	??	Generalized shuffle theorem
$\tilde{H}_\lambda^{(s)}(X;q,t)$	??	??

Thank you!

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Schiffmann to Shuffle

- Shuffle algebra S given by the image of Laurent polynomials $\phi \in \Bbbk[x_1^{\pm 1}, \dots, x_l^{\pm 1}]$ via map

$$H_{q,t} \colon \phi \mapsto \sum_{w \in S_l} w \left(\frac{\phi \prod_{i<j}(1 - qtx_i/x_j)}{\prod_{i<j}((1 - x_j/x_i)(1 - qx_i/x_j)(1 - tx_i/x_j))} \right)$$

- (Schiffmann-Vasserot, 2013) There exists isomorphism $\psi \colon S \to \mathcal{E}^+$.
- (Negut, 2014) gives well-defined

$$D_{b_1,\dots,b_l} = \psi \left(H_{q,t} \left(\frac{x_1^{b_1} \cdots x_l^{b_l}}{\prod_{i=1}^{l-1}(1 - qtx_i/x_{i+1})} \right) \right)$$

Key Relationship (BHMPS, 2023)

For $\zeta = \psi(\phi) \in \mathcal{E}^+$,

$$\omega(\zeta \cdot 1) = \omega \, \mathrm{pol}_X \, H_{q,t}(\phi) \, .$$

Cauchy Identity

- (Twisted) non-symmetric Hall-Littlewood polynomials $E^{\sigma}_{\lambda}(x; q)$ defined via Demazure-Lusztig operators.

$$T_i = q s_i + (1-q)\frac{s_i - 1}{1 - x_{i+1}/x_i}$$

- Dual basis F^{σ}_{λ}.

Cauchy identity

$$\frac{\prod_{i<j}(1 - q\,t\,x_i\,y_j)}{\prod_{i\leq j}(1 - t\,x_i\,y_j)} = \sum_{\mathbf{a}\geq 0} t^{|\mathbf{a}|}\,E^{\sigma}_{\mathbf{a}}(x_1,\ldots,x_l; q^{-1})\,F^{\sigma}_{\mathbf{a}}(y_1,\ldots,y_l; q),$$

- $\mathcal{L}_{\beta/\alpha} = H_q(w_0(F^{\sigma^{-1}}_{\beta}(x;q)\overline{E^{\sigma^{-1}}_{\alpha}(x;q)}))$