1. INTRODUCTION

First, we recall some definitions.

1.1. Definition. An \( N \times N \) matrix \( U \) is unitary if \( UU^* = I_N \) where \( U^* \) is the conjugate transpose of \( U \). Then, \( U(N) \) is the compact Lie group of all \( N \times N \) unitary matrices. Since \( U(N-1) \hookrightarrow U(N) \) via a canonical embedding, we also define

\[
U(\infty) := \bigcup_{N=1}^{\infty} U(N)
\]

that is, \( U(\infty) \) are all infinite \( \mathbb{N} \times \mathbb{N} \) unitary matrices that differ from the identity matrix only in a fixed number of positions.

1.2. Definition. A normalized character of \( U(N) \) is a function \( \chi: U(N) \to \mathbb{C} \) such that

(a) \( \chi(e) = 1 \) (normalized),
(b) \( \chi(ab) = \chi(ba) \) (constant on conjugacy classes),
(c) \( \left( \sum c_i \chi(a_i) \right) \left( \sum c_j \chi(a_j) \right) = \sum c_i c_j \chi(a_i a_j^{-1}) \geq 0 \) (nonnegative definite),
(d) \( \chi \) is continuous.

Normalized characters form a convex set since \( t \chi_1 + (1-t) \chi_2 \) meets all the axioms of a normalized character for all \( t \in [0,1] \). Then, we can discuss the following notion.

1.3. Definition. An extreme character \( \chi: U(N) \to \mathbb{C} \) is a normalized character such that \( \chi \neq t \chi_1 + (1-t) \chi_2 \) for any \( t \in (0,1) \) for normalized characters \( \chi_1, \chi_2 \neq \chi \).

1.4. Definition. The \( N \)-dimensional torus is

\[
\mathbb{T}^N := \{ (x_1, \ldots, x_N) \in \mathbb{C}^N \mid |x_i| = 1 \}
\]

and lies in \( U(N) \) as diagonal matrices. The finitary torus is \( \mathbb{T}_{\text{fin}}^\infty := \bigcup_{N=1}^{\infty} \mathbb{T}^N \).

Recall one of our main goals is to understand the following theorem.
1.5. **Theorem** (Edrei-Voiculescu). Extreme characters of $U(\infty)$ are functions $\chi : T^\infty_{fin} \to \mathbb{C}$ depending on countably many parameters

$$\begin{cases}
\alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0);\\
\beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0);\\
\gamma^\pm \geq 0
\end{cases}$$

such that

$$\sum_i \alpha_i^+ + \sum_i \alpha_i^- + \sum_i \beta_i^+ + \sum_i \beta_i^- < \infty, \quad \beta_1^+ + \beta_1^- \leq 1$$

Furthermore, these functions have the form

$$\chi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_1, x_2, \ldots) = \prod_{j=1}^{\infty} \Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_j)$$

where $\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm} : \mathbb{T} \to \mathbb{C}$ is the continuous function

$$\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x) := e^{\gamma^+(x-1)+\gamma^-(x^{-1}-1)} \prod_{i=1}^{\infty} \left( \frac{1 + \beta_i^+(x-1)}{1 - \alpha_i^+(x-1)} , \frac{1 + \beta_i^-(x^{-1}-1)}{1 - \alpha_i^-(x^{-1}-1)} \right).$$

1.6. **Goal.** In this presentation, we will outline two very special examples of this parameterization, namely when

(a) $\beta^+ = (\beta, 0, 0, \ldots), \beta^- = \alpha^\pm = (0, 0, \ldots), \gamma^\pm = 0$ for $\beta \in [0, 1]$ so that

$$\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x) = 1 + \beta(x-1) \implies \chi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_1, x_2, \ldots) = \prod_{j=1}^{\infty} (1 + \beta(x_j - 1))$$

(b) $\alpha^+ = (\alpha, 0, 0, \ldots), \beta^\pm = \alpha^\pm = (0, 0, \ldots), \gamma^\pm = 0$ for $\alpha \in [0, 1]$ so that

$$\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x) = \frac{1}{1 - \alpha(x-1)} \implies \chi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_1, x_2, \ldots) = \prod_{j=1}^{\infty} \frac{1}{1 - \alpha(x_j - 1)}$$

2. **Symmetric Functions**

In the last lecture, we introduced the following.

2.1. **Definition.** Given a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, the **Schur polynomial** is given by

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det(x_j^{\lambda_i+N-i})_{i,j=1}^{N}}{\det(x_j^{-i})_{i,j=1}^{N}}$$

Also, if $\lambda$ has $\lambda_N \geq 0$, we can use “Littlewood’s Combinatorial Description” of Schur functions

2.2. **Proposition.** Given a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0,$

$$s_\lambda(x_1, \ldots, x_N) = \sum_{T \in \text{SSYT}(\lambda)} x^{\text{wt}(T)}$$
where $x^{\text{wt}(T)} = \prod_{j=1}^{\sum \lambda_i} x_j^\#$ of $j$’s in $T$.

2.3. Example.

\[ s_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2 \]

\[
\begin{array}{c}
1 \\
2
\end{array}
+ \begin{array}{c}
1 \\
2
\end{array}
\]

We also proved that

2.4. **Theorem.** The irreducible representations of $U(N)$ are in one-to-one correspondence with $\{ \lambda \in \mathbb{Z}^N \mid \lambda_1 \geq \cdots \geq \lambda_N \}$ where the character of representation $T_\lambda$ of $U(N)$ corresponding to $\lambda$ has character given by

\[
\text{Tr} \left( T_\lambda \left( \begin{array}{c}
x_1 \\
m_{2} \\
\vdots \\
x_N
\end{array} \right) \right) = s_\lambda(x_1, \ldots x_N)
\]

We will work with two special cases of the Schur polynomials.

2.5. **Definition.** Let $e_m(x_1, \ldots, x_N) := s_{(1^m)}(x_1, \ldots, x_N)$ be the elementary symmetric polynomials.

2.6. **Example.** Using the semistandard Young tableaux formula for Schur functions (Littlewood’s combinatorial description), we compute

(a)

\[ e_2(x_1, x_2) = x_1 x_2 \]

\[
\begin{array}{c}
1 \\
2
\end{array}
\]

(b)

\[ e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3 \]

\[
\begin{array}{c}
1 \\
2 \\
3
\end{array}
+ \begin{array}{c}
1 \\
3
\end{array}
+ \begin{array}{c}
2 \\
3
\end{array}
\]

(c)

\[ e_3(x_1, x_2, x_3) = x_1 x_2 x_3 \]

\[
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\]

2.7. **Remark.** $e_N(x_1, \ldots, x_N)$ encodes character of the “determinant representation” of $U(N)$, that is

\[ T(U)v = (\det U)v = x_1 x_2 \cdots x_N v \]
since the determinant is just the product of the eigenvalues. More generally, 
\[ e_m(x_1, \ldots, x_N) \] 
codes the representation induced by the \( U(N) \)-action on 
\( \bigwedge^m \mathbb{C}^N \):
\[
U \cdot (v_1 \wedge \cdots \wedge v_m) = (Uv_1 \wedge \cdots \wedge Uv_m)
\]

Importantly, we also compute, generalizing our example above

2.8. **Proposition.** For \( 0 < m \leq n \),
\[
e_m(x_1, x_2, \ldots, x_n) = \sum_{T \in \text{SSYT}((1^m)) \text{ filled with elements of } \{1, \ldots, n\}} x^{\text{wt}(T)} = \sum_{I \subseteq \{1, \ldots, n\}, |I| = m} x^I
\]
where \( x^I := \prod_{i \in I} x_i \) and consequently,
\[
e_m(1, \ldots, 1) = \binom{n}{m}
\]

**Proof.** To see this, we simply observe that a single column semistandard tableau with \( m \) rows filled with letters \( \{1, \ldots, n\} \) is a choice of \( m \) distinct elements of \( \{1, \ldots, n\} \) since columns must be strictly increasing. \( \square \)

2.9. **Definition.** Let \( h_m(x_1, \ldots, x_N) := s_m(x_1, \ldots, x_N) \) be the complete homogeneous symmetric polynomials.

2.10. **Example.** Using again our tableaux formula for Schur functions, we compute

(a)
\[
h_2(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2
\]
\[
\begin{array}{c}
1
\end{array} + \begin{array}{c}
1
\quad \begin{array}{c}
2
\end{array}
\end{array} + \begin{array}{c}
2
\quad \begin{array}{c}
2
\end{array}
\end{array}
\]

(b)
\[
h_2(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2
\]
\[
\begin{array}{c}
1
\quad \begin{array}{c}
1
\quad \begin{array}{c}
2
\quad \begin{array}{c}
3
\end{array}
\end{array}
\end{array}
\end{array} + \begin{array}{c}
1
\quad \begin{array}{c}
3
\end{array}
\end{array} + \begin{array}{c}
2
\quad \begin{array}{c}
3
\end{array}
\end{array} + \begin{array}{c}
3
\quad \begin{array}{c}
3
\end{array}
\end{array}
\]

2.11. **Proposition.** For \( 0 < m \leq n \),
\[
h_m(x_1, x_2, \ldots, x_n) = \sum_{T \in \text{SSYT}((m)) \text{ filled with elements of } \{1, \ldots, n\}} x^{\text{wt}(T)} = \sum_{I \text{ multiset of } \{1, \ldots, n\}, |I| = m} x^I
\]
where \( x^I := \prod_{i \in I} x_i \) and consequently,
\[
h_m(1, \ldots, 1)
\]
\[
= \text{Number of ways to choose a multiset of size } m \text{ from } n \text{ things}
\]
\[
= \binom{n + m - 1}{m} = \binom{n + m - 1}{n - 1}
\]
2.12. **Remark.** The combinatorics of the identity above follow by considering a “stars and bars” approach, namely, both expressions are in bijection with the number of ways to place \( n - 1 \) bars among \( m \) stars, allowing bars to be consecutive with each other.

\[
\{1, 1, 1, 2, 4, 5\} \rightarrow \ast \ast \ast | \ast | \ast |
\]

2.13. **Definition.** Let

\[
\binom{n}{m} := \binom{n + m - 1}{m}
\]

be the number of ways to choose a multiset of size \( m \) from \( n \) things.

3. **Two Examples of \( U(\infty) \) characters**

Now, we wish to take a sequence of \( U(N) \) characters to get a character of \( U(\infty) \).

3.1. **Definition.** We say that a sequence of central functions \( f_N \) (i.e. \( f_N \) only depends on the eigenvalues of the input) on \( U(N) \) converge to a central function \( f \) on \( U(\infty) \) if, for every fixed \( K \), we have

\[
f_N(x_1, \ldots, x_K, 1, 1, \ldots, 1) \to f(x_1, \ldots, x_K, 1, 1, \ldots)
\]

uniformly on the \( K \)-torus \( T^K \) of diagonal matrices.

3.2. **Proposition.** Let \( L : \mathbb{N} \to \mathbb{N} \) be a sequence such that \( L(N)/N \to \beta \in [0, 1] \) as \( N \to \infty \). Then,

\[
\frac{e_{L(N)}(x_1, \ldots, x_N)}{e_{L(N)}(1, \ldots, 1)} \to \prod_{i=1}^{\infty} (1 + \beta(x_i - 1)), \quad (x_1, x_2, \ldots) \in T^\infty_{fin}
\]

**Proof.** Fix \( K \leq N \). Then,

\[
e_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1)
= \sum_{T \in SSYT((1^{L(N)})) \text{ labelled with } \{1, \ldots, N\}} x^{\text{wt}(T) \leq K}
= \sum_{\text{binary } K \text{ sequences } \epsilon} \# \{N \text{ sequences with sum } L(N) \text{ that start with } (\epsilon_1, \ldots, \epsilon_K)\} \epsilon^{(\epsilon_1, \ldots, \epsilon_K)}
= \sum_{\text{binary } K \text{ sequences } \epsilon} \binom{N - K}{L(N) - \sum_{i=1}^K \epsilon_i} x^{\epsilon_1} \cdots x^{\epsilon_K}
\]

where the last equality comes from considering how to fill tableaux of the form

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3.4. Proposition. \( \text{This approach.} \) derive the proposition directly from this observation. See [Pet12] §4.1.10 for

\[
\frac{e_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1)}{e_{L(N)}(1, \ldots, 1)} = \sum_{\text{binary } K \text{ sequences } \epsilon} \left( \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right)^{x_1^\epsilon_1 \cdots x_K^\epsilon_K}
\]

where

\[
\left( \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right)^{x_1^\epsilon_1 \cdots x_K^\epsilon_K} = \frac{(N - K)!}{N!} \times \frac{(L(N))!}{(L(N) - \sum_{i=1}^{K} \epsilon_i)!} \times \frac{(N - L(N))!}{(N - L(N) - (K - \sum_{i=1}^{K} \epsilon_i))!}
\]

Thus, taking the limit as \( N \to \infty \) on our ratio, we get

\[
\sum_{\text{binary } K \text{ sequences } \epsilon} x_1^\epsilon_1 \cdots x_K^\epsilon_K \beta^{\sum_{i=1}^{K} \epsilon_i} (1 - \beta)^{K - \sum_{i=1}^{K} \epsilon_i} = \prod_{i=1}^{K} ((1 - \beta) + \beta x_i)
\]

and so, taking \( K \to \infty \) completes the proof. \( \square \)

3.3. Remark. An astute reader may notice that \((1 - \beta)^{K - \sum_{i=1}^{K} \epsilon_i} \beta^{\sum_{i=1}^{K} \epsilon_i}\) represents the probability of \(\sum_{i=1}^{K} \epsilon_i\) successes in \( K \) trials where each attempt has probability of success \( \beta \). One can use “de Finetti’s theorem” in order to derive the proposition directly from this observation. See [Pet12] §4.1.10 for this approach.

3.4. Proposition. Let \( L: \mathbb{N} \to \mathbb{N} \) be a sequence such that \( L(N)/N \to \alpha \in [0, 1] \) as \( N \to \infty \). Then,

\[
h_{L(N)}(x_1, \ldots, x_N) \to \prod_{i=1}^{\infty} \frac{1}{1 - \alpha(x_i - 1)}, \quad (x_1, x_2, \ldots) \in \mathbb{T}_{\text{fin}}^\infty
\]

Proof. We proceed much as in the proposition above. For a fixed \( K \leq N \), we have

\[
h_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1) \]

\[
= \sum_{\epsilon \in \mathbb{N}_0^K} \#\{N \text{ sequences with sum } L(N) \text{ starting with } (\epsilon_1, \ldots, \epsilon_K)\} x_1^\epsilon_1 \cdots x_K^\epsilon_K
\]

\[
= \sum_{\epsilon \in \mathbb{N}_0^K} \left( \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right)^{x_1^\epsilon_1 \cdots x_K^\epsilon_K}
\]

where the last line comes from thinking about

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Fill \( \sum_{i=1}^{K} \epsilon_i \) boxes with \( \{1, \ldots, K\} \)

Fill \( N - \sum_{i=1}^{K} \epsilon_i \) boxes with \( \{K+1, \ldots, N\} \)
and so
\[
\frac{h_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1)}{h_{L(N)}(1, \ldots, 1)} = \sum_{\epsilon \in \mathbb{N}_0^K} \left[ \left( \frac{N - K}{L(N) - \sum_{i=1}^K \epsilon_i} \right) / \left( \frac{N}{L(N)} \right) \right] x_1^{\epsilon_1} \ldots x_K^{\epsilon_K}
\]
Consider that, for fixed \( K \leq N \), we have
\[
\left( \frac{N - K}{L(N) - \sum_{i=1}^K \epsilon_i} \right) / \left( \frac{N}{L(N)} \right) = \frac{(N + L(N) - K - \sum \epsilon_i - 1)!}{(N + L(N) - 1)!} \times \frac{(L(N))!}{(L(N) - \sum \epsilon_i)!} \times \frac{(N - 1)!}{(N - K - 1)!}
\]
\[
\approx \frac{(L(N))^{\sum \epsilon_i} N^K}{(N + L(N))^{K + \sum \epsilon_i}}
\]
\[
= \left( \frac{L(N)}{N} \right)^{\sum \epsilon_i} \left( \frac{1}{1 + \frac{L(N)}{N}} \right)^{K + \sum \epsilon_i}
\]
\[
N \to \infty \quad \left( \frac{\alpha}{1 + \alpha} \right)^{\sum \epsilon_i} \left( \frac{1}{1 + \alpha} \right)^K
\]
Thus,
\[
\lim_{N \to \infty} \frac{h_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1)}{h_{L(N)}(1, \ldots, 1)} = \sum_{\epsilon} \left( \frac{1}{1 + \alpha} \right)^K \left( \frac{\alpha}{1 + \alpha} \right)^{\sum \epsilon_i} x_1^{\epsilon_1} \ldots x_K^{\epsilon_K}
\]
\[
= \prod_{i=1}^K \left( \frac{1}{1 + \alpha} \right) \left( 1 + \frac{\alpha}{1 + \alpha} x_i + \left( \frac{\alpha}{1 + \alpha} \right)^2 x_i^2 + \cdots \right)
\]
\[
= \prod_{i=1}^K \frac{1}{1 + \alpha - \alpha x_i}
\]
So, taking \( K \to \infty \) completes the proof. \( \square \)

References