TWO ELEMENTARY EXAMPLES OF EXTREME CHARACTERS OF $U(\infty)$ INTEGRABLE PROBABILITY READING SEMINAR

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1. INTRODUCTION

First, we recall some definitions.

1.1. Definition. An $N \times N$ matrix $U$ is unitary if $UU^* = I_N$ where $U^*$ is the conjugate transpose of $U$. Then, $U(N)$ is the compact Lie group of all $N \times N$ unitary matrices. Since $U(N-1) \hookrightarrow U(N)$ via a canonical embedding, we also define

$$U(\infty) := \bigcup_{N=1}^{\infty} U(N)$$

that is, $U(\infty)$ are all infinite $N \times N$ unitary matrices that differ from the identity matrix only in a fixed number of positions.

1.2. Definition. A normalized character of $U(N)$ is a function $\chi: U(N) \rightarrow \mathbb{C}$ such that

(a) $\chi(e) = 1$ (normalized),
(b) $\chi(ab) = \chi(ba)$ (constant on conjugacy classes),
(c) $(\sum c_i \chi(a_i))(\sum c_j \chi(a_j)) = \sum c_i c_j \chi(a_i a_j^{-1}) \geq 0$ (nonnegative definite),
(d) $\chi$ is continuous.

Normalized characters form a convex set since $t\chi_1 + (1-t)\chi_2$ meets all the axioms of a normalized character for all $t \in [0, 1]$. Then, we can discuss the following notion.

1.3. Definition. An extreme character $\chi: U(N) \rightarrow \mathbb{C}$ is a normalized character such that $\chi \neq t\chi_1 + (1-t)\chi_2$ for any $t \in (0, 1)$ for normalized characters $\chi_1, \chi_2 \neq \chi$.

1.4. Definition. The $N$-dimensional torus is

$$\mathbb{T}^N := \{(x_1, \ldots, x_N) \in \mathbb{C}^N \mid |x_i| = 1\}$$

and lies in $U(N)$ as diagonal matrices. The finitary torus is $\mathbb{T}^\infty_{fin} := \bigcup_{N=1}^{\infty} \mathbb{T}^N$.

Recall one of our main goals is to understand the following theorem.
1.5. **Theorem** (Edrei-Voiculescu). Extreme characters of $U(\infty)$ are functions $\chi : T_\infty \to \mathbb{C}$ depending on countably many parameters

\[
\begin{cases}
\alpha^+ = (\alpha_1^+ \geq \alpha_2^+ \geq \cdots \geq 0); \\
\beta^+ = (\beta_1^+ \geq \beta_2^+ \geq \cdots \geq 0); \\
\gamma^+ \geq 0
\end{cases}
\]

such that

\[
\sum_i \alpha_i^+ + \sum_i \alpha_i^- + \sum_i \beta_i^+ + \sum_i \beta_i^- < \infty, \quad \beta_1^+ + \beta_1^- \leq 1
\]

Furthermore, these functions have the form

\[
\chi_{\alpha^+, \beta^+, \gamma^+}(x_1, x_2, \ldots) = \prod_{j=1}^\infty \Phi_{\alpha^+, \beta^+, \gamma^+}(x_j)
\]

where $\Phi_{\alpha^+, \beta^+, \gamma^+} : T \to \mathbb{C}$ is the continuous function

\[
\Phi_{\alpha^+, \beta^+, \gamma^+}(x) := e^{\gamma^+(x-1)+\gamma^-(x^{-1}-1)} \prod_{i=1}^\infty \left( \frac{1 + \beta_i^+(x-1)}{1 - \alpha_i^+(x-1)} \cdot \frac{1 + \beta_i^-(x^{-1}-1)}{1 - \alpha_i^-(x^{-1}-1)} \right).
\]

1.6. **Goal.** In this presentation, we will outline two very special examples of this parameterization, namely when

(a) $\beta^+ = (\beta, 0, 0, \ldots), \beta^- = \alpha^+ = (0, 0, \ldots), \gamma^+ = 0$ for $\beta \in [0, 1]$ so that

$\Phi_{\alpha^+, \beta^+, \gamma^+}(x) = 1 + \beta(x-1) \implies \chi_{\alpha^+, \beta^+, \gamma^+}(x_1, x_2, \ldots) = \prod_{j=1}^\infty (1 + \beta(x_j - 1))$

(b) $\alpha^+ = (\alpha, 0, 0, \ldots), \beta^+ = \alpha^+ = (0, 0, \ldots), \gamma^+ = 0$ for $\alpha \in [0, 1]$ so that

$\Phi_{\alpha^+, \beta^+, \gamma^+}(x) = \frac{1}{1 - \alpha(x-1)} \implies \chi_{\alpha^+, \beta^+, \gamma^+}(x_1, x_2, \ldots) = \prod_{j=1}^\infty \frac{1}{1 - \alpha(x_j - 1)}$

2. Symmetric Functions

In the last lecture, we introduced the following.

2.1. **Definition.** Given a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, the **Schur polynomial** is given by

\[
s_\lambda(x_1, \ldots, x_N) = \frac{\det(x_j^{\lambda_i + N-i})_{i,j=1}^N}{\det(x_j^{N-i})_{i,j=1}^N}
\]

Also, if $\lambda$ has $\lambda_N \geq 0$, we can use “Littlewood’s Combinatorial Description” of Schur functions

2.2. **Proposition.** Given a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$,

\[
s_\lambda(x_1, \ldots, x_N) = \sum_{T \in SSYT(\lambda)} x^{\text{wt}(T)}
\]
where \( x^{\text{wt}(T)} = \prod_{j=1}^{\sum \lambda_i} x_j^{\# \text{of } j \text{'s in } T} \).

2.3. Example.

\[
s_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2
\]

\[
\begin{array}{c}
1 & 1 \\
2 & \\
\end{array} +
\begin{array}{c}
1 & 2 \\
2 & \\
\end{array}
\]

We also proved that

2.4. Theorem. The irreducible representations of \( U(N) \) are in one-to-one correspondence with \( \{ \lambda \in \mathbb{Z}^N \mid \lambda_1 \geq \cdots \geq \lambda_N \} \) where the character of representation \( T_\lambda \) of \( U(N) \) corresponding to \( \lambda \) has character given by

\[
\text{Tr} \left( T_\lambda \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \right) = s_\lambda(x_1, \ldots, x_N)
\]

We will work with two special cases of the Schur polynomials.

2.5. Definition. Let \( e_m(x_1, \ldots, x_N) := s_{(1^m)}(x_1, \ldots, x_N) \) be the elementary symmetric polynomials.

2.6. Example. Using the semistandard Young tableaux formula for Schur functions (Littlewood’s combinatorial description), we compute

(a)

\[
e_2(x_1, x_2) = x_1 x_2
\]

\[
\begin{array}{c}
1 \\
2 \\
\end{array}
\]

(b)

\[
e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3
\]

\[
\begin{array}{c}
1 & 1 & 2 \\
2 & 3 & 3 \\
\end{array}
\]

(c)

\[
e_3(x_1, x_2, x_3) = x_1 x_2 x_3
\]

\[
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\]

2.7. Remark. \( e_N(x_1, \ldots, x_N) \) encodes character of the “determinant representation” of \( U(N) \), that is

\[
T(U)v = (\text{det } U)v = x_1 x_2 \cdots x_N v
\]
since the determinant is just the product of the eigenvalues. More generally, $e_m(x_1, \ldots, x_N)$ encodes the representation induced by the $U(N)$-action on $\wedge^m \mathbb{C}^N$:

$$U \cdot (v_1 \wedge \cdots \wedge v_m) = (Uv_1 \wedge \cdots \wedge Uv_m)$$

Importantly, we also compute, generalizing our example above

2.8. Proposition. For $0 < m \leq n$,

$$e_m(x_1, x_2, \ldots, x_n) = \sum_{T \in \text{SSYT}((1^m)) \text{ filled with elements of } \{1, \ldots, n\}} \prod_{i \in I} x_i$$

where $x^I := \prod_{i \in I} x_i$ and consequently,

$$e_m(1, \ldots, 1) = \binom{n}{m}$$

Proof. To see this, we simply observe that a single column semistandard tableau with $m$ rows filled with letters $\{1, \ldots, n\}$ is a choice of $m$ distinct elements of $\{1, \ldots, n\}$ since columns must be strictly increasing. $\square$

2.9. Definition. Let $h_m(x_1, \ldots, x_N) := s_m(x_1, \ldots, x_N)$ be the complete homogeneous symmetric polynomials.

2.10. Example. Using again our tableaux formula for Schur functions, we compute

(a) $h_2(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$

\[
\begin{array}{cccc}
1 & 1 & & \\
1 & 2 & & \\
2 & & & \\
\end{array}
\]

(b) $h_2(x_1, x_2, x_3) = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$

\[
\begin{array}{cccccc}
1 & 1 & & & & \\
1 & 2 & & & & \\
1 & 3 & & & & \\
2 & 2 & & & & \\
2 & 3 & & & & \\
3 & 3 & & & & \\
\end{array}
\]

2.11. Proposition. For $0 < m \leq n$,

$$h_m(x_1, x_2, \ldots, x_n) = \sum_{T \in \text{SSYT}((m)) \text{ filled with elements of } \{1, \ldots, n\}} \prod_{i \in I} x_i$$

where $x^I := \prod_{i \in I} x_i$ and consequently,

$$h_m(1, \ldots, 1) = \binom{n + m - 1}{m} = \binom{n + m - 1}{n - 1}$$
2.12. Remark. The combinatorics of the identity above follow by considering a “stars and bars” approach, namely, both expressions are in bijection with the number of ways to place \( n - 1 \) bars among \( m \) stars, allowing bars to be consecutive with each other.

\[
\{1, 1, 1, 2, 4, 5\} \rightarrow \ast \ast \ast | \ast | | | \ast
\]

2.13. Definition. Let

\[
\left( \begin{array}{c} n \\ m \end{array} \right) := \left( \begin{array}{c} n + m - 1 \\ m \end{array} \right)
\]

be the number of ways to choose a multiset of size \( m \) from \( n \) things.

3. Two Examples of \( U(\infty) \) characters

Now, we wish to take a sequence of \( U(N) \) characters to get a character of \( U(\infty) \).

3.1. Definition. We say that a sequence of central functions \( f_N \) (i.e. \( f_N \) only depends on the eigenvalues of the input) on \( U(N) \) converge to a central function \( f \) on \( U(\infty) \) if, for every fixed \( K \), we have

\[
f_N(x_1, \ldots, x_K, 1, 1, \ldots, 1) \to f(x_1, \ldots, x_K, 1, 1, \ldots)
\]

uniformly on the \( K \)-torus \( \mathbb{T}^K \) of diagonal matrices.

3.2. Proposition. Let \( L: \mathbb{N} \to \mathbb{N} \) be a sequence such that \( L(N)/N \to \beta \in [0, 1] \) as \( N \to \infty \). Then,

\[
\frac{e_{L(N)}(x_1, \ldots, x_N)}{e_{L(N)}(1, \ldots, 1)} \to \prod_{i=1}^{\infty} \left( 1 + \beta(x_i - 1) \right), \quad (x_1, x_2, \ldots) \in \mathbb{T}_f^{\infty}
\]

Proof. Fix \( K \leq N \). Then,

\[
e_{L(N)}(x_1, \ldots, x_K, 1, 1, \ldots, 1)
\]

\[
= \sum_{T \in \text{SSYT}(1 \uparrow L(N))} x^{\text{wt}(T) \leq K}
\]

\[
= \sum_{\text{binary } K \text{ sequences } \epsilon} \# \{ N \text{ sequences with sum } L(N) \text{ that start with } (\epsilon_1, \ldots, \epsilon_K) \} x^{\epsilon_1, \ldots, \epsilon_K}
\]

\[
= \sum_{\text{binary } K \text{ sequences } \epsilon} \left( \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right) x_{\epsilon_1} \cdots x_{\epsilon_K}
\]

where the last equality comes from considering how to fill tableaux of the form
\[
\begin{align*}
\text{Fill } \sum_{i=1}^{K} \epsilon_i \text{ boxes with } \{1, \ldots, K\} \\
\text{Fill } L(N) - \sum_{i=1}^{K} \epsilon_i \text{ boxes with } \{K+1, \ldots, N\}
\end{align*}
\]

\[\frac{e_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1)}{e_{L(N)}(1, \ldots, 1)} = \sum_{\text{binary } K \text{ sequences } \epsilon} \left( \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right) x_1^{\epsilon_1} \cdots x_K^{\epsilon_K}
\]

where
\[
\left( \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right) = \frac{(N - K)!}{N!} \times \frac{(L(N))!}{(L(N) - \sum_{i=1}^{K} \epsilon_i)!} \times \frac{(N - L(N))!}{(N - L(N) - (K - \sum_{i=1}^{K} \epsilon_i))!}
\]

Thus, taking the limit as \( N \to \infty \) on our ratio, we get
\[
\sum_{\text{binary } K \text{ sequences } \epsilon} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \beta^{\sum_{i=1}^{K} \epsilon_i} (1 - \beta)^{K - \sum_{i=1}^{K} \epsilon_i} = \prod_{i=1}^{K} ((1 - \beta) + \beta x_i)
\]
and so, taking \( K \to \infty \) completes the proof. \qed

3.3. **Remark.** An astute reader may notice that \((1 - \beta)^{K - \sum_{i=1}^{K} \epsilon_i} \beta^{\sum_{i=1}^{K} \epsilon_i}\) represents the probability of \(\sum_{i=1}^{K} \epsilon_i\) successes in \(K\) trials where each attempt has probability of success \(\beta\). One can use “de Finetti’s theorem” in order to derive the proposition directly from this observation. See [Pet12]§4.1.10 for this approach.

3.4. **Proposition.** Let \( L : \mathbb{N} \to \mathbb{N} \) be a sequence such that \( L(N)/N \to \alpha \in [0, 1] \) as \( N \to \infty \). Then,
\[
\frac{h_{L(N)}(x_1, \ldots, x_N)}{h_{L(N)}(1, \ldots, 1)} \to \prod_{i=1}^{\infty} \frac{1}{1 - \alpha(x_i - 1)}, \quad (x_1, x_2, \ldots) \in \mathbb{T}_{fin}^{\infty}
\]

**Proof.** We proceed much as in the proposition above. For a fixed \( K \leq N \), we have
\[
\begin{align*}
&h_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1) \\
&= \sum_{\epsilon \in \mathbb{N}_0^K} \# \{ N \text{ sequences with sum } L(N) \text{ starting with } (\epsilon_1, \ldots, \epsilon_K) \} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \\
&= \sum_{\epsilon \in \mathbb{N}_0^K} \left( \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right) x_1^{\epsilon_1} \cdots x_K^{\epsilon_K}
\end{align*}
\]
where the last line comes from thinking about
and so
\[
\frac{h_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1)}{h_{L(N)}(1, \ldots, 1)} = \sum_{\epsilon \in \mathbb{N}_0^K} \frac{\left( \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right)}{\left( \frac{N}{L(N)} \right)} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K}
\]

Consider that, for fixed \( K \leq N \), we have
\[
\left( \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right) / \left( \frac{N}{L(N)} \right) = \frac{(N + L(N) - K - \sum_{i=1}^{K} \epsilon_i - 1)!}{(N + L(N) - 1)!} \times \frac{(L(N))!}{(L(N) - \sum_{i=1}^{K} \epsilon_i)!} \times \frac{(N - 1)!}{(N - K - 1)!}
\]
\[
\approx \frac{(L(N))^{\sum_{i=1}^{K} \epsilon_i} N^K}{(N + L(N))^{K + \sum_{i=1}^{K} \epsilon_i}}
\]
\[
= \left( \frac{L(N)}{N} \right)^{\sum_{i=1}^{K} \epsilon_i} \left( \frac{1}{1 + \frac{L(N)}{N}} \right)^{K + \sum_{i=1}^{K} \epsilon_i}
\]
\[
N \rightarrow \infty \left( \frac{\alpha}{1 + \alpha} \right)^{\sum_{i=1}^{K} \epsilon_i} \left( \frac{1}{1 + \alpha} \right)^K
\]

Thus,
\[
\lim_{N \rightarrow \infty} \frac{h_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1)}{h_{L(N)}(1, \ldots, 1)} = \sum_{\epsilon} \left( \frac{1}{1 + \alpha} \right)^K \left( \frac{\alpha}{1 + \alpha} \right)^{\sum_{i=1}^{K} \epsilon_i} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K}
\]
\[
= \prod_{i=1}^{K} \left( \frac{1}{1 + \alpha} \right) \left( 1 + \frac{\alpha}{1 + \alpha} x_i + \left( \frac{\alpha}{1 + \alpha} \right)^2 x_i^2 + \cdots \right)
\]
\[
= \prod_{i=1}^{K} \frac{1}{1 + \alpha} \times \frac{1}{1 - \frac{\alpha}{1 + \alpha} x_i}
\]
\[
= \prod_{i=1}^{K} \frac{1}{1 + \alpha - \alpha x_i}
\]

So, taking \( K \rightarrow \infty \) completes the proof. \( \square \)

References