TWO ELEMENTARY EXAMPLES OF EXTREME CHARACTERS OF $U(\infty)$
INTEGRABLE PROBABILITY READING SEMINAR

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1. INTRODUCTION

First, we recall some definitions.

1.1. Definition. An $N \times N$ matrix $U$ is unitary if $UU^* = I_N$ where $U^*$ is the conjugate transpose of $U$. Then, $U(N)$ is the compact Lie group of all $N \times N$ unitary matrices. Since $U(N-1) \hookrightarrow U(N)$ via a canonical embedding, we also define

$$U(\infty) := \bigcup_{N=1}^{\infty} U(N)$$

that is, $U(\infty)$ are all infinite $\mathbb{N} \times \mathbb{N}$ unitary matrices that differ from the identity matrix only in a fixed number of positions.

1.2. Definition. A normalized character of $U(N)$ is a function $\chi: U(N) \to \mathbb{C}$ such that

(a) $\chi(e) = 1$ (normalized),
(b) $\chi(ab) = \chi(ba)$ (constant on conjugacy classes),
(c) $(\sum c_i \chi(a_i))(\sum c_j \chi(a_j)) = \sum c_i c_j \chi(a_i a_j^{-1}) \geq 0$ (nonnegative definite),
(d) $\chi$ is continuous.

Normalized characters form a convex set since $t\chi_1 + (1-t)\chi_2$ meets all the axioms of a normalized character for all $t \in [0,1]$. Then, we can discuss the following notion.

1.3. Definition. An extreme character $\chi: U(N) \to \mathbb{C}$ is a normalized character such that $\chi \neq t\chi_1 + (1-t)\chi_2$ for any $t \in (0,1)$ for normalized characters $\chi_1, \chi_2 \neq \chi$.

1.4. Definition. The $N$-dimensional torus is

$$\mathbb{T}^N := \{(x_1, \ldots, x_N) \in \mathbb{C}^N \mid |x_i| = 1\}$$

and lies in $U(N)$ as diagonal matrices. The finitary torus is $\mathbb{T}_{fin}^\infty := \bigcup_{N=1}^{\infty} \mathbb{T}^N$.

Recall one of our main goals is to understand the following theorem.

\hspace{1cm} \textit{Date: February 1, 2019.}
1.5. **Theorem** (Edrei-Voiculescu). Extreme characters of $U(\infty)$ are functions $\chi: T^\infty_{fin} \to \mathbb{C}$ depending on countably many parameters

\[
\begin{cases}
\alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0); \\
\beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0); \\
\gamma^\pm \geq 0
\end{cases}
\]

such that

\[
\sum_i \alpha_i^+ + \sum_i \alpha_i^- + \sum_i \beta_i^+ + \sum_i \beta_i^- < \infty, \quad \beta_1^+ + \beta_1^- \leq 1
\]

Furthermore, these functions have the form

\[
\chi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_1, x_2, \ldots) = \prod_{j=1}^{\infty} \Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_j)
\]

where $\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}: \mathbb{T} \to \mathbb{C}$ is the continuous function

\[
\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x) := e^{\gamma^+(x-1)+\gamma^-(x^{-1}-1)} \prod_{i=1}^{\infty} \left( \frac{1 + \beta_i^+(x-1)}{1 - \alpha_i^+(x-1)} \cdot \frac{1 + \beta_i^-(x^{-1}-1)}{1 - \alpha_i^-(x^{-1}-1)} \right).
\]

1.6. **Goal.** In this presentation, we will outline two very special examples of this parameterization, namely when

(a) $\beta^+ = (\beta, 0, 0, \ldots), \beta^- = \alpha^\pm = (0, 0, \ldots), \gamma^\pm = 0$ for $\beta \in [0, 1]$ so that

\[
\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x) = 1 + \beta(x-1) \implies \chi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_1, x_2, \ldots) = \prod_{j=1}^{\infty} (1 + \beta(x_j - 1))
\]

(b) $\alpha^+ = (\alpha, 0, 0, \ldots), \beta^\pm = \alpha^\pm = (0, 0, \ldots), \gamma^\pm = 0$ for $\alpha \in [0, 1]$ so that

\[
\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x) = \frac{1}{1 - \alpha(x-1)} \implies \chi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_1, x_2, \ldots) = \prod_{j=1}^{\infty} \frac{1}{1 - \alpha(x_j - 1)}
\]

### 2. Symmetric Functions

In the last lecture, we introduced the following.

2.1. **Definition.** Given a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, the **Schur polynomial** is given by

\[
s_\lambda(x_1, \ldots, x_N) = \frac{\det(x_j^{\lambda_i+N-i})_{i,j=1}^{N}}{\det(x_j^{N-i})_{i,j=1}^{N}}
\]

Also, if $\lambda$ has $\lambda_N \geq 0$, we can use “Littlewood’s Combinatorial Description” of Schur functions

2.2. **Proposition.** Given a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$, 

\[
s_\lambda(x_1, \ldots, x_N) = \sum_{T \in \text{SSYT}(\lambda)} x^{\text{wt}(T)}
\]
where \( x^{\text{wt}(T)} = \prod_{j=1}^{\sum \lambda_i} x_j^{\# \text{j’s in } T} \).

2.3. Example.

\[
s_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2
\]

We also proved that

2.4. Theorem. The irreducible representations of \( U(N) \) are in one-to-one correspondence with \( \{ \lambda \in \mathbb{Z}^N \mid \lambda_1 \geq \cdots \geq \lambda_N \} \) where the character of representation \( T_\lambda \) of \( U(N) \) corresponding to \( \lambda \) has character given by

\[
\text{Tr} \left( T_\lambda \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \right) = s_\lambda(x_1, \ldots, x_N)
\]

We will work with two special cases of the Schur polynomials.

2.5. Definition. Let \( e_m(x_1, \ldots, x_N) := s_{(1^m)}(x_1, \ldots, x_N) \) be the elementary symmetric polynomials.

2.6. Example. Using the semistandard Young tableaux formula for Schur functions (Littlewood's combinatorial description), we compute

(a)

\[
e_2(x_1, x_2) = x_1 x_2
\]

(b)

\[
e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3
\]

(c)

\[
e_3(x_1, x_2, x_3) = x_1 x_2 x_3
\]

2.7. Remark. \( e_N(x_1, \ldots, x_N) \) encodes character of the “determinant representation” of \( U(N) \), that is

\[
T(U)v = (\det U)v = x_1 x_2 \cdots x_N v
\]
since the determinant is just the product of the eigenvalues. More generally, \(e_m(x_1, \ldots, x_N)\) encodes the representation induced by the \(U(N)\)-action on \(\wedge^m \mathbb{C}^N\):

\[
U \cdot (v_1 \wedge \cdots \wedge v_m) = (U v_1 \wedge \cdots \wedge U v_m)
\]

Importantly, we also compute, generalizing our example above

2.8. **Proposition.** For \(0 < m \leq n\),

\[
e_m(x_1, x_2, \ldots, x_n) = \sum_{T \in SSYT((1^m)) \text{ filled with elements of } \{1, \ldots, n\}} x^{wt(T)} = \sum_{I \subseteq \{1, \ldots, n\}, |I| = m} x^I
\]

where \(x^I := \prod_{i \in I} x_i\) and consequently,

\[
e_m(1, \ldots, 1) = \binom{n}{m}
\]

**Proof.** To see this, we simply observe that a single column semistandard tableau with \(m\) rows filled with letters \(\{1, \ldots, n\}\) is a choice of \(m\) distinct elements of \(\{1, \ldots, n\}\) since columns must be strictly increasing. \(\square\)

2.9. **Definition.** Let \(h_m(x_1, \ldots, x_N) := s_m(x_1, \ldots, x_N)\) be the complete homogeneous symmetric polynomials.

2.10. **Example.** Using again our tableaux formula for Schur functions, we compute

(a)

\[
h_2(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2
\]

\[
\begin{array}{ccc}
1 & 1 & \\
1 & 2 & \\
2 & 2 & \\
\end{array}
\]

(b)

\[
h_2(x_1, x_2, x_3) = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2
\]

\[
\begin{array}{ccc}
1 & 1 & \\
1 & 2 & \\
1 & 3 & \\
2 & 2 & \\
2 & 3 & \\
3 & 3 & \\
\end{array}
\]

2.11. **Proposition.** For \(0 < m \leq n\),

\[
h_m(x_1, x_2, \ldots, x_n) = \sum_{T \in SSYT((m)) \text{ filled with elements of } \{1, \ldots, n\}} x^{wt(T)} = \sum_{I \text{ multiset of } \{1, \ldots, n\}, |I| = m} x^I
\]

where \(x^I := \prod_{i \in I} x_i\) and consequently,

\[
h_m(1, \ldots, 1)
\]

\[
= \text{Number of ways to choose a multiset of size } m \text{ from } n \text{ things}
\]

\[
= \binom{n + m - 1}{m} = \binom{n + m - 1}{n - 1}
\]
2.12. Remark. The combinatorics of the identity above follow by considering a “stars and bars” approach, namely, both expressions are in bijection with the number of ways to place \( n - 1 \) bars among \( m \) stars, allowing bars to be consecutive with each other.

\[ \{1, 1, 1, 2, 4, 5\} \rightarrow ***|* || *\]

2.13. Definition. Let

\[ \binom{n}{m} := \binom{n + m - 1}{m} \]

be the number of ways to choose a multiset of size \( m \) from \( n \) things.

3. Two Examples of \( U(\infty) \) characters

Now, we wish to take a sequence of \( U(N) \) characters to get a character of \( U(\infty) \).

3.1. Definition. We say that a sequence of central functions \( f_N \) (i.e. \( f_N \) only depends on the eigenvalues of the input) on \( U(N) \) converge to a central function \( f \) on \( U(\infty) \) if, for every fixed \( K \), we have

\[ f_N(x_1, \ldots, x_K, 1, 1, \ldots, 1) \rightarrow f(x_1, \ldots, x_K, 1, 1, \ldots) \]

uniformly on the \( K \)-torus \( \mathbb{T}^K \) of diagonal matrices.

3.2. Proposition. Let \( L: \mathbb{N} \rightarrow \mathbb{N} \) be a sequence such that \( L(N)/N \rightarrow \beta \in [0, 1] \) as \( N \rightarrow \infty \). Then,

\[ \frac{e^{L(N)}(x_1, \ldots, x_N)}{e^{L(N)}(1, \ldots, 1)} \rightarrow \prod_{i=1}^\infty (1 + \beta(x_i - 1)), \quad (x_1, x_2, \ldots) \in \mathbb{T}^\infty_{fin} \]

Proof. Fix \( K \leq N \). Then,

\[
e^{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1) = \sum_{T \in \text{SSYT}(\{1^{L(N)}\}) \text{ labelled with } \{1, \ldots, N\}} x^{\text{wt}(T) \leq K} = \sum_{\text{binary } K \text{ sequences } \epsilon} \# \{ N \text{ sequences with sum } L(N) \text{ that start with } (\epsilon_1, \ldots, \epsilon_K) \} x^{(\epsilon_1, \ldots, \epsilon_K)} = \sum_{\text{binary } K \text{ sequences } \epsilon} \left( \frac{N - K}{L(N) - \sum_{i=1}^K \epsilon_i} \right) x^{\epsilon_1} x^\epsilon_K \]

where the last equality comes from considering how to fill tableaux of the form
Proposition. 3.4. Derive the proposition directly from this observation. See [Pet12]§4.1.10 for

Then, taking

Thus, taking the limit as $N \to \infty$ on our ratio, we get

and so, taking $K \to \infty$ completes the proof. □

3.3. Remark. An astute reader may notice that $(1-\beta)^{K-\sum_{i=1}^{K} \epsilon_{i}} \beta^{\sum_{i=1}^{K} \epsilon_{i}}$ represents the probability of $\sum_{i=1}^{K} \epsilon_{i}$ successes in $K$ trials where each attempt has probability of success $\beta$. One can use “de Finetti’s theorem” in order to derive the proposition directly from this observation. See [Pet12]§4.1.10 for this approach.

3.4. Proposition. Let $L: \mathbb{N} \to \mathbb{N}$ be a sequence such that $L(N)/N \to \alpha \in [0, 1]$ as $N \to \infty$. Then,

Proof. We proceed much as in the proposition above. For a fixed $K \leq N$, we have

where the last line comes from thinking about
and so
\[
\frac{h_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1)}{h_{L(N)}(1, \ldots, 1)} = \sum_{\epsilon \in \mathbb{N}_0^K} \left[ \left( \frac{N - K}{L(N) - \sum_{i=1}^K \epsilon_i} \right) / \left( \frac{N}{L(N)} \right) \right] x_1^{\epsilon_1} \cdots x_K^{\epsilon_K}
\]

Consider that, for fixed \( K \leq N \), we have
\[
\left( \frac{N - K}{L(N) - \sum_{i=1}^K \epsilon_i} \right) / \left( \frac{N}{L(N)} \right) = \left( \frac{N + L(N) - \sum_{i=1}^K \epsilon_i - 1}{N + L(N) - 1} \right) \times \frac{(L(N))!}{(L(N) - \sum \epsilon_i)!} \times \frac{(N - 1)!}{(N - K - 1)!}
\]
\[
\approx \frac{(L(N))^\sum \epsilon_i N^K}{(N + L(N))^{K + \sum \epsilon_i}}
\]
\[
\approx \left( \frac{L(N)}{N} \right)^\sum \epsilon_i \left( \frac{1}{1 + \frac{L(N)}{N}} \right)^{K + \sum \epsilon_i}
\]
\[
N \to \infty \quad \left( \frac{\alpha}{1 + \alpha} \right)^{\sum \epsilon_i} \left( \frac{1}{1 + \alpha} \right)^K
\]

Thus,
\[
\lim_{N \to \infty} \frac{h_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1)}{h_{L(N)}(1, \ldots, 1)} = \sum_{\epsilon} \left( \frac{1}{1 + \alpha} \right)^K \left( \frac{\alpha}{1 + \alpha} \right)^{\sum \epsilon_i} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K}
\]
\[
= \prod_{i=1}^K \left( \frac{1}{1 + \alpha} \right) \left( 1 + \frac{\alpha}{1 + \alpha} x_i + \left( \frac{\alpha}{1 + \alpha} \right)^2 x_i^2 + \cdots \right)
\]
\[
= \prod_{i=1}^K \frac{1}{1 + \alpha - \alpha x_i}
\]

So, taking \( K \to \infty \) completes the proof. \( \square \)

**References**