TWO ELEMENTARY EXAMPLES OF EXTREME CHARACTERS OF $U(\infty)$ INTEGRABLE PROBABILITY READING SEMINAR

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1. INTRODUCTION

First, we recall some definitions.

1.1. Definition. An $N \times N$ matrix $U$ is unitary if $UU^* = I_N$ where $U^*$ is the conjugate transpose of $U$. Then, $U(N)$ is the compact Lie group of all $N \times N$ unitary matrices. Since $U(N - 1) \hookrightarrow U(N)$ via a canonical embedding, we also define

$$U(\infty) := \bigcup_{N=1}^{\infty} U(N)$$

that is, $U(\infty)$ are all infinite $N \times N$ unitary matrices that differ from the identity matrix only in a fixed number of positions.

1.2. Definition. A normalized character of $U(N)$ is a function $\chi: U(N) \to \mathbb{C}$ such that

- (a) $\chi(e) = 1$ (normalized),
- (b) $\chi(ab) = \chi(ba)$ (constant on conjugacy classes),
- (c) $(\sum c_i \chi(a_i)) (\sum c_j \chi(a_j)) = \sum c_i c_j \chi(a_i a_j^{-1}) \geq 0$ (nonnegative definite),
- (d) $\chi$ is continuous.

Normalized characters form a convex set since $t\chi_1 + (1-t)\chi_2$ meets all the axioms of a normalized character for all $t \in [0,1]$. Then, we can discuss the following notion.

1.3. Definition. An extreme character $\chi: U(N) \to \mathbb{C}$ is a normalized character such that $\chi \neq t\chi_1 + (1-t)\chi_2$ for any $t \in (0,1)$ for normalized characters $\chi_1, \chi_2 \neq \chi$.

1.4. Definition. The $N$-dimensional torus is

$$\mathbb{T}^N := \{(x_1, \ldots, x_N) \in \mathbb{C}^N | |x_i| = 1\}$$

and lies in $U(N)$ as diagonal matrices. The finitary torus is $\mathbb{T}^\infty_{fin} := \bigcup_{N=1}^{\infty} \mathbb{T}^N$.  

Recall one of our main goals is to understand the following theorem.

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1.5. **Theorem** (Edrei-Voiculescu). *Extreme characters of* $U(\infty)$ *are functions* $\chi: T^\infty_{fin} \to \mathbb{C}$ *depending on countably many parameters*

\[
\begin{align*}
\alpha^\pm &= (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0); \\
\beta^\pm &= (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0); \\
\gamma^\pm &\geq 0
\end{align*}
\]

such that

\[
\sum_i \alpha_i^+ + \sum_i \alpha_i^- + \sum_i \beta_i^+ + \sum_i \beta_i^- < \infty, \quad \beta_1^++\beta_1^- \leq 1
\]

Furthermore, these functions have the form

\[
\chi_{\alpha^+, \beta^+, \gamma^+}(x_1, x_2, \ldots) = \prod_{j=1}^{\infty} \Phi_{\alpha^+, \beta^+, \gamma^+}(x_j)
\]

where $\Phi_{\alpha^+, \beta^+, \gamma^+}: \mathbb{T} \to \mathbb{C}$ is the continuous function

\[
\Phi_{\alpha^+, \beta^+, \gamma^+}(x) := e^{\gamma^+(x-1)+\gamma^-(x^{-1}-1) \sum_{i=1}^{\infty} \left( \frac{1 + \beta_i^+(x-1)}{1 - \alpha_i^+(x-1)} \cdot \frac{1 + \beta_i^-(x^{-1}-1)}{1 - \alpha_i^-(x^{-1}-1)} \right)}
\]

1.6. **Goal.** In this presentation, we will outline two very special examples of this parameterization, namely when

(a) $\beta^+ = (\beta, 0, 0, \ldots), \beta^- = \alpha^+ = (0, 0, \ldots), \gamma^+ = 0$ for $\beta \in [0, 1]$ so that

$\Phi_{\alpha^+, \beta^+, \gamma^+}(x) = 1 + \beta(x-1) \implies \chi_{\alpha^+, \beta^+, \gamma^+}(x_1, x_2, \ldots) = \prod_{j=1}^{\infty} (1 + \beta(x_j - 1))$

(b) $\alpha^+ = (\alpha, 0, 0, \ldots), \beta^\pm = \alpha^+ = (0, 0, \ldots), \gamma^+ = 0$ for $\alpha \in [0, 1]$ so that

$\Phi_{\alpha^+, \beta^+, \gamma^+}(x) = \frac{1}{1 - \alpha(x-1)} \implies \chi_{\alpha^+, \beta^+, \gamma^+}(x_1, x_2, \ldots) = \prod_{j=1}^{\infty} \frac{1}{1 - \alpha(x_j - 1)}$

2. **Symmetric Functions**

In the last lecture, we introduced the following.

2.1. **Definition.** Given a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, the *Schur polynomial* is given by

\[
s_\lambda(x_1, \ldots, x_N) = \frac{\det(x_j^{\lambda_i+N-i})_{i,j=1}^N}{\det(x_j^{N-i})_{i,j=1}^N}
\]

Also, if $\lambda$ has $\lambda_N \geq 0$, we can use “Littlewood’s Combinatorial Description” of Schur functions

2.2. **Proposition.** Given a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$,

\[
s_\lambda(x_1, \ldots, x_N) = \sum_{T \in \text{SSYT}(\lambda)} x^{\text{wt}(T)}
\]
where $x^{\text{wt}(T)} = \prod_{j=1}^{\sum \lambda_i} x_j^j$ of $j$’s in $T$.

2.3. Example.

$$s_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$$

\[\begin{array}{cc}
1 & 1 \\
2 & \\
\end{array} \quad + \quad \begin{array}{cc}
1 & 2 \\
2 & \\
\end{array}\]

We also proved that

2.4. Theorem. The irreducible representations of $U(N)$ are in one-to-one correspondence with $\{\lambda \in \mathbb{Z}^N \mid \lambda_1 \geq \cdots \geq \lambda_N\}$ where the character of representation $T_{\lambda}$ of $U(N)$ corresponding to $\lambda$ has character given by

$$\text{Tr} \left( T_{\lambda} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \right) = s_{\lambda}(x_1, \ldots, x_N)$$

We will work with two special cases of the Schur polynomials.

2.5. Definition. Let $e_m(x_1, \ldots, x_N) := s_{(1^m)}(x_1, \ldots, x_N)$ be the elementary symmetric polynomials.

2.6. Example. Using the semistandard Young tableaux formula for Schur functions (Littlewood’s combinatorial description), we compute

(a) $$e_2(x_1, x_2) = x_1 x_2$$

\[\begin{array}{c}
1 \\
2 \\
\end{array}\]

(b) $$e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3$$

\[\begin{array}{c}
1 \\
2 \\
\end{array} + \begin{array}{c}
1 \\
3 \\
\end{array} + \begin{array}{c}
2 \\
3 \\
\end{array}\]

(c) $$e_3(x_1, x_2, x_3) = x_1 x_2 x_3$$

\[\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}\]

2.7. Remark. $e_N(x_1, \ldots, x_N)$ encodes character of the “determinant representation” of $U(N)$, that is

$$T(U)v = (\det U)v = x_1 x_2 \cdots x_N v$$
since the determinant is just the product of the eigenvalues. More generally,
\( e_m(x_1, \ldots, x_N) \) encodes the representation induced by the \( U(N) \)-action on \( \bigwedge^m \mathbb{C}^N \):
\[
U \cdot (v_1 \wedge \cdots \wedge v_m) = (Uv_1 \wedge \cdots \wedge Uv_m)
\]

Importantly, we also compute, generalizing our example above

**2.8. Proposition.** For \( 0 < m \leq n \),
\[
e_m(x_1, x_2, \ldots, x_n) = \sum_{T \in \text{SSYT}((1^m)) \text{ filled with elements of } \{1, \ldots, n\}} x^{\text{wt}(T)} = \sum_{I \subseteq \{1, \ldots, n\}, |I| = m} x^I
\]
where \( x^I := \prod_{i \in I} x_i \) and consequently,
\[
e_m(\underbrace{1, \ldots, 1}_{n}) = \binom{n}{m}
\]

**Proof.** To see this, we simply observe that a single column semistandard tableau with \( m \) rows filled with letters \( \{1, \ldots, n\} \) is a choice of \( m \) distinct elements of \( \{1, \ldots, n\} \) since columns must be strictly increasing. \( \square \)

**2.9. Definition.** Let \( h_m(x_1, \ldots, x_N) := s_m(x_1, \ldots, x_N) \) be the complete homogeneous symmetric polynomials.

**2.10. Example.** Using again our tableaux formula for Schur functions, we compute

(a)
\[
h_2(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2
\]
\[
\begin{array}{c}
1 & 1 \\
1 & 2 \\
2 & 2
\end{array}
\]

(b)
\[
h_2(x_1, x_2, x_3) = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2
\]
\[
\begin{array}{c,c,c}
1 & 1 & 1 \\
1 & 2 & 2 \\
1 & 3 & 3 \\
2 & 2 & 2 \\
2 & 3 & 3 \\
3 & 3 & 3
\end{array}
\]

**2.11. Proposition.** For \( 0 < m \leq n \),
\[
h_m(x_1, x_2, \ldots, x_n) = \sum_{T \in \text{SSYT}((m)) \text{ filled with elements of } \{1, \ldots, n\}} x^{\text{wt}(T)} = \sum_{I \text{ multiset of } \{1, \ldots, n\}, |I| = m} x^I
\]
where \( x^I := \prod_{i \in I} x_i \) and consequently,
\[
h_m(\underbrace{1, \ldots, 1}_{n})
\]
\[
= \text{Number of ways to choose a multiset of size } m \text{ from } n \text{ things}
\]
\[
= \binom{n + m - 1}{m} = \binom{n + m - 1}{n - 1}
\]
2.12. Remark. The combinatorics of the identity above follow by considering a “stars and bars” approach, namely, both expressions are in bijection with the number of ways to place \( n - 1 \) bars among \( m \) stars, allowing bars to be consecutive with each other.

\[
\{1, 1, 1, 2, 4, 5\} \rightarrow \star \star \star | \star | \star | \star
\]

2.13. Definition. Let

\[
\binom{n}{m} := \binom{n + m - 1}{m}
\]

be the number of ways to choose a multiset of size \( m \) from \( n \) things.

3. Two Examples of \( U(\infty) \) characters

Now, we wish to take a sequence of \( U(N) \) characters to get a character of \( U(\infty) \).

3.1. Definition. We say that a sequence of central functions \( f_N \) (i.e. \( f_N \) only depends on the eigenvalues of the input) on \( U(N) \) converge to a central function \( f \) on \( U(\infty) \) if, for every fixed \( K \), we have

\[
f_N(x_1, \ldots, x_K, 1, 1, \ldots) \rightarrow f(x_1, \ldots, x_K, 1, 1, \ldots)
\]

uniformly on the \( K \)-torus \( \mathbb{T}^K \) of diagonal matrices.

3.2. Proposition. Let \( L: \mathbb{N} \rightarrow \mathbb{N} \) be a sequence such that \( L(N)/N \rightarrow \beta \in [0, 1] \) as \( N \rightarrow \infty \). Then,

\[
\frac{e_{L(N)}(x_1, \ldots, x_N)}{e_{L(N)}(1, \ldots, 1)} \rightarrow \prod_{i=1}^{\infty} (1 + \beta(x_i - 1)), \quad (x_1, x_2, \ldots) \in \mathbb{T}_f^\infty
\]

Proof. Fix \( K \leq N \). Then,

\[
e_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1)
= \sum_{T \in \text{SSYT}(L(N)) \text{ labelled with } \{1, \ldots, N\}} x^\text{wt}(T|\leq K)
= \sum_{\text{binary } K \text{ sequences } \epsilon} \# \{N \text{ sequences with sum } L(N) \text{ that start with } (\epsilon_1, \ldots, \epsilon_K)\} x^{\epsilon_1 \ldots \epsilon_K}
= \sum_{\text{binary } K \text{ sequences } \epsilon} \left( \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right) x^{\epsilon_1 \ldots \epsilon_K}
\]

where the last equality comes from considering how to fill tableaux of the form
Proposition.

3.4. We proceed much as in the proposition above. For a fixed $K \leq N$, we have

\[
\frac{h_{L(N)}(x_1, \ldots, x_N)}{h_{L(N)}(1, \ldots, 1)} \to \prod_{i=1}^{\infty} \frac{1}{1 - \alpha(x_i - 1)}, \quad (x_1, x_2, \ldots) \in \mathbb{T}_{fin}^\infty
\]

Proof. We proceed much as in the proposition above. For a fixed $K \leq N$, we have

\[
h_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1) = \sum_{\epsilon \in H_0^K} \#\{N \text{ sequences with sum } L(N) \text{ starting with } (\epsilon_1, \ldots, \epsilon_K)\} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K}
\]

\[
= \sum_{\epsilon \in H_0^K} \left( \left( \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right) \right) x_1^{\epsilon_1} \cdots x_K^{\epsilon_K}
\]

where the last line comes from thinking about

\[
\text{Fill } \sum_{i=1}^{K} \epsilon_i \text{ boxes with } \{1, \ldots, K\} \quad \text{Fill } N - \sum_{i=1}^{K} \epsilon_i \text{ boxes with } \{K + 1, \ldots, N\}
\]

3.3. Remark. An astute reader may notice that $(1 - \beta)^{K - \sum_{i=1}^{K} \epsilon_i} \beta^{\sum_{i=1}^{K} \epsilon_i}$ represents the probability of $\sum_{i=1}^{K} \epsilon_i$ successes in $K$ trials where each attempt has probability of success $\beta$. One can use “de Finetti’s theorem” in order to derive the proposition directly from this observation. See [Pet12]\S4.1.10 for this approach.

3.4. Proposition. Let $L: \mathbb{N} \to \mathbb{N}$ be a sequence such that $L(N)/N \to \alpha \in [0, 1]$ as $N \to \infty$. Then,

\[
\frac{h_{L(N)}(x_1, \ldots, x_N)}{h_{L(N)}(1, \ldots, 1)} \to \prod_{i=1}^{\infty} \frac{1}{1 - \alpha(x_i - 1)}, \quad (x_1, x_2, \ldots) \in \mathbb{T}_{fin}^\infty
\]
and so
\[
\frac{h_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1)}{h_{L(N)}(1, \ldots, 1)} = \sum_{\epsilon \in N^K} \left[ \left( \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right) / \left( \frac{N}{L(N)} \right) \right] x_1^{\epsilon_1} \cdots x_K^{\epsilon_K}
\]

Consider that, for fixed \( K \leq N \), we have
\[
\left( \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right) / \left( \frac{N}{L(N)} \right) = \left( \frac{N + L(N) - K - \sum_{i=1}^{K} \epsilon_i}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right) \times \frac{(L(N))!}{(N + L(N) - 1)!} \times \frac{(N - K - 1)!}{(N - L(N) - 1)!} \approx \frac{(L(N))^{\sum_{i=1}^{K} \epsilon_i}}{(N + L(N))^{K + \sum_{i=1}^{K} \epsilon_i}} \left( \frac{1}{1 + \frac{L(N)}{N}} \right)^{K + \sum_{i=1}^{K} \epsilon_i}
\]

Thus,
\[
\lim_{N \to \infty} \frac{h_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1)}{h_{L(N)}(1, \ldots, 1)} = \sum_{\epsilon} \left( \frac{1}{1 + \alpha} \right)^K \left( \frac{\alpha}{1 + \alpha} \right)^{\sum_{i=1}^{K} \epsilon_i} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K}
\]
\[
= \prod_{i=1}^{K} \left( \frac{1}{1 + \alpha} \right) \left( 1 + \frac{\alpha}{1 + \alpha} x_i + \left( \frac{\alpha}{1 + \alpha} \right)^2 x_i^2 + \cdots \right)
\]
\[
= \prod_{i=1}^{K} \frac{1}{1 + \alpha - \alpha x_i}
\]

So, taking \( K \to \infty \) completes the proof. \( \square \)

References