1. Introduction

First, we recall some definitions.

1.1. Definition. An $N \times N$ matrix $U$ is unitary if $UU^* = I_N$ where $U^*$ is the conjugate transpose of $U$. Then, $U(N)$ is the compact Lie group of all $N \times N$ unitary matrices. Since $U(N - 1) \hookrightarrow U(N)$ via a canonical embedding, we also define

$$U(\infty) := \bigcup_{N=1}^{\infty} U(N)$$

that is, $U(\infty)$ are all infinite $\mathbb{N} \times \mathbb{N}$ unitary matrices that differ from the identity matrix only in a fixed number of positions.

1.2. Definition. A normalized character of $U(N)$ is a function $\chi: U(N) \to \mathbb{C}$ such that

(a) $\chi(e) = 1$ (normalized),
(b) $\chi(ab) = \chi(ba)$ (constant on conjugacy classes),
(c) $\left( \sum c_i \chi(a_i) \right) \left( \sum c_j \chi(a_j) \right) = \sum c_i c_j \chi(a_i a_j^{-1}) \geq 0$ (nonnegative definite),
(d) $\chi$ is continuous.

Normalized characters form a convex set since $t\chi_1 + (1-t)\chi_2$ meets all the axioms of a normalized character for all $t \in [0,1]$. Then, we can discuss the following notion.

1.3. Definition. An extreme character $\chi: U(N) \to \mathbb{C}$ is a normalized character such that $\chi \neq t\chi_1 + (1-t)\chi_2$ for any $t \in (0,1)$ for normalized characters $\chi_1, \chi_2 \neq \chi$.

1.4. Definition. The $N$-dimensional torus is

$$\mathbb{T}^N := \{(x_1, \ldots, x_N) \in \mathbb{C}^N \mid |x_i| = 1\}$$

and lies in $U(N)$ as diagonal matrices. The finitary torus is $\mathbb{T}_{fin}^{\infty} := \bigcup_{N=1}^{\infty} \mathbb{T}^N$.

Recall one of our main goals is to understand the following theorem.

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1.5. Theorem (Edrei-Voiculescu). Extreme characters of $U(\infty)$ are functions $\chi: T_{fin}^\infty \rightarrow \mathbb{C}$ depending on countably many parameters

\[
\begin{cases}
\alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0);
\beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0);
\gamma^\pm \geq 0
\end{cases}
\]

such that

\[
\sum_i \alpha_i^+ + \sum_i \alpha_i^- + \sum_i \beta_i^+ + \sum_i \beta_i^- < \infty, \quad \beta_1^+ + \beta_1^- \leq 1
\]

Furthermore, these functions have the form

\[
\chi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_1, x_2, \ldots) = \prod_{j=1}^\infty \Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_j)
\]

where $\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}: T \rightarrow \mathbb{C}$ is the continuous function

\[
\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x) := e^{\gamma^+(x-1) + \gamma^-(x^{-1}-1)} \prod_{i=1}^\infty \left( \frac{1 + \beta_i^+(x-1)}{1 - \alpha_i^+(x-1)} \cdot \frac{1 + \beta_i^-(x^{-1}-1)}{1 - \alpha_i^-(x^{-1}-1)} \right).
\]

1.6. Goal. In this presentation, we will outline two very special examples of this parameterization, namely when

(a) $\beta^+ = (\beta, 0, 0, \ldots), \beta^- = \alpha^\pm = (0, 0, \ldots), \gamma^\pm = 0$ for $\beta \in [0, 1]$ so that

\[
\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x) = 1 + \beta(x-1) \implies \chi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_1, x_2, \ldots) = \prod_{j=1}^\infty (1 + \beta(x_j - 1))
\]

(b) $\alpha^+ = (\alpha, 0, 0, \ldots), \beta^\pm = \alpha^\pm = (0, 0, \ldots), \gamma^\pm = 0$ for $\alpha \in [0, 1]$ so that

\[
\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x) = \frac{1}{1 - \alpha(x-1)} \implies \chi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_1, x_2, \ldots) = \prod_{j=1}^\infty \frac{1}{1 - \alpha(x_j - 1)}
\]

2. Symmetric Functions

In the last lecture, we introduced the following.

2.1. Definition. Given a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, the Schur polynomial is given by

\[
s_\lambda(x_1, \ldots, x_N) = \frac{\det(x_j^{\lambda_i+N-i})_{i,j=1}^N}{\det(x_j^{N-i})_{i,j=1}^N}
\]

Also, if $\lambda$ has $\lambda_N \geq 0$, we can use “Littlewood’s Combinatorial Description” of Schur functions

2.2. Proposition. Given a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$,

\[
s_\lambda(x_1, \ldots, x_N) = \sum_{T \in \text{SSYT}(\lambda)} x^{\text{wt}(T)}
\]
where \( x^{\text{wt}(T)} = \prod_{j=1}^{\sum \lambda_j} x_j^j \) of \( j \)'s in \( T \).

2.3. Example.

\[
\begin{align*}
\text{s}_{(2,1)}(x_1, x_2) &= x_1^2 x_2 + x_1 x_2^2 \\
&= x_1 \begin{array}{c} 1 \\ 2 \end{array} + x_2 \begin{array}{c} 1 \\ 2 \end{array}
\end{align*}
\]

We also proved that

2.4. Theorem. The irreducible representations of \( U(N) \) are in one-to-one correspondence with \( \{ \lambda \in \mathbb{Z}^N \mid \lambda_1 \geq \cdots \geq \lambda_N \} \) where the character of representation \( T_\lambda \) of \( U(N) \) corresponding to \( \lambda \) has character given by

\[
\text{Tr} \left( T_\lambda \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \right) = s_\lambda(x_1, \ldots, x_N)
\]

We will work with two special cases of the Schur polynomials.

2.5. Definition. Let \( e_m(x_1, \ldots, x_N) := s_{(1^m)}(x_1, \ldots, x_N) \) be the elementary symmetric polynomials.

2.6. Example. Using the semistandard Young tableaux formula for Schur functions (Littlewood’s combinatorial description), we compute

(a) \[
e_2(x_1, x_2) = x_1 x_2
\]

\[
\begin{array}{c} 1 \\ 2 \end{array}
\]

(b) \[
e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3
\]

\[
\begin{array}{c} 1 \\ 2 \end{array} + \begin{array}{c} 1 \\ 3 \end{array} + \begin{array}{c} 2 \\ 3 \end{array}
\]

(c) \[
e_3(x_1, x_2, x_3) = x_1 x_2 x_3
\]

\[
\begin{array}{c} 1 \\ 2 \\ 3 \end{array}
\]

2.7. Remark. \( e_N(x_1, \ldots, x_N) \) encodes character of the “determinant representation” of \( U(N) \), that is

\[
T(U)v = (\det U)v = x_1 x_2 \cdots x_N v
\]
since the determinant is just the product of the eigenvalues. More generally, 
\( e_m(x_1,\ldots,x_N) \) encodes the representation induced by the \( U(N) \)-action on \( \bigwedge^m \mathbb{C}^N \):

\[
U \cdot (v_1 \wedge \cdots \wedge v_m) = (Uv_1 \wedge \cdots \wedge Uv_m)
\]

Importantly, we also compute, generalizing our example above

2.8. **Proposition.** For \( 0 < m \leq n \),

\[
e_m(x_1,x_2,\ldots,x_n) = \sum_{T \in \text{SSYT}((1^m)) \text{ filled with elements of } \{1,\ldots,n\}} x^{\text{wt}(T)} = \sum_{I \subseteq \{1,\ldots,n\}, |I|=m} x^I
\]

where \( x^I := \prod_{i \in I} x_i \) and consequently,

\[
e_m(1,\ldots,1) = \binom{n}{m}
\]

**Proof.** To see this, we simply observe that a single column semistandard tableau with \( m \) rows filled with letters \( \{1,\ldots,n\} \) is a choice of \( m \) distinct elements of \( \{1,\ldots,n\} \) since columns must be strictly increasing. \( \square \)

2.9. **Definition.** Let \( h_m(x_1,\ldots,x_N) := s_m(x_1,\ldots,x_N) \) be the complete homogeneous symmetric polynomials.

2.10. **Example.** Using again our tableaux formula for Schur functions, we compute

(a) \( \quad h_2(x_1,x_2) = x_1^2 + x_1 x_2 + x_2^2 \)

(b) \( \quad h_2(x_1,x_2,x_3) = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2 \)

2.11. **Proposition.** For \( 0 < m \leq n \),

\[
h_m(x_1,x_2,\ldots,x_n) = \sum_{T \in \text{SSYT}(m) \text{ filled with elements of } \{1,\ldots,n\}} x^{\text{wt}(T)} = \sum_{I \text{ multiset of } \{1,\ldots,n\}, |I|=m} x^I
\]

where \( x^I := \prod_{i \in I} x_i \) and consequently,

\[
h_m(1,\ldots,1)
\]

\( \quad = \text{Number of ways to choose a multiset of size } m \text{ from } n \text{ things} \)

\[
= \binom{n + m - 1}{m} = \binom{n + m - 1}{n-1}
\]
2.12. **Remark.** The combinatorics of the identity above follow by considering a “stars and bars” approach, namely, both expressions are in bijection with the number of ways to place $n - 1$ bars among $m$ stars, allowing bars to be consecutive with each other.

\[
\{1, 1, 1, 2, 4, 5\} \rightarrow \star \star \star | \star \star \star
\]

2.13. **Definition.** Let \[
\binom{n}{m} := \binom{n + m - 1}{m}
\]
be the number of ways to choose a multiset of size $m$ from $n$ things.

### 3. Two Examples of $U(\infty)$ characters

Now, we wish to take a sequence of $U(N)$ characters to get a character of $U(\infty)$.

#### 3.1. **Definition.** We say that a sequence of central functions $f_N$ (i.e. $f_N$ only depends on the eigenvalues of the input) on $U(N)$ converge to a central function $f$ on $U(\infty)$ if, for every fixed $K$, we have

\[f_N(x_1, \ldots, x_K, 1, 1, \ldots) \to f(x_1, \ldots, x_K, 1, 1, \ldots)\]

uniformly on the $K$-torus $T^K$ of diagonal matrices.

#### 3.2. **Proposition.** Let $L: \mathbb{N} \to \mathbb{N}$ be a sequence such that $L(N)/N \to \beta \in [0, 1]$ as $N \to \infty$. Then,

\[\frac{e_{L(N)}(x_1, \ldots, x_N)}{e_{L(N)}(1, \ldots, 1)} \to \prod_{i=1}^{\infty} (1 + \beta(x_i - 1)), \quad (x_1, x_2, \ldots) \in T^\infty_{fin}\]

**Proof.** Fix $K \leq N$. Then,

\[e_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1) = \sum_{T \in SSYT((1^{L(N)}) \text{ labelled with } \{1, \ldots, N\})} x^{\text{wt}(T)} \leq K\]

\[= \sum_{\text{binary } K \text{ sequences } \epsilon} \#\{N \text{ sequences with sum } L(N) \text{ that start with } (\epsilon_1, \ldots, \epsilon_K)\} x^{\epsilon_1, \ldots, \epsilon_K}\]

\[= \sum_{\text{binary } K \text{ sequences } \epsilon} \left(\frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i}\right) x^{\epsilon_1} \cdots x^{\epsilon_K}\]

where the last equality comes from considering how to fill tableaux of the form

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Proposition.

3.4. This approach. derive the proposition directly from this observation. See [Pet12]§4.1.10 for

3.3. Let

\[ e_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1) = \sum_{\text{binary } K \text{ sequences } \epsilon} \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} = \left( \frac{N - K}{(L(N) - \sum_{i=1}^{K} \epsilon_i)} \right) x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \]

where

\[ \left( \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right) \left( \frac{N}{L(N)} \right) = \frac{(N - K)!}{N!} \times \frac{(L(N))!}{(L(N) - \sum_{i=1}^{K} \epsilon_i)!} \times \frac{(N - L(N))!}{(N - L(N) - (K - \sum_{i=1}^{K} \epsilon_i))!} \]

Thus, taking the limit as \( N \to \infty \) on our ratio, we get

\[ \sum_{\text{binary } K \text{ sequences } \epsilon} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \beta^{\sum_{i=1}^{K} \epsilon_i} = \prod_{i=1}^{K} ((1 - \beta) + \beta x_i) \]

and so, taking \( K \to \infty \) completes the proof.

\[ \square \]

3.3. Remark. An astute reader may notice that \( (1 - \beta)^{K - \sum_{i=1}^{K} \epsilon_i} \beta^{\sum_{i=1}^{K} \epsilon_i} \) represents the probability of \( \sum_{i=1}^{K} \epsilon_i \) successes in \( K \) trials where each attempt has probability of success \( \beta \). One can use “de Finetti’s theorem” in order to derive the proposition directly from this observation. See [Pet12]§4.1.10 for this approach.

3.4. Proposition. Let \( L: \mathbb{N} \to \mathbb{N} \) be a sequence such that \( L(N)/N \to \alpha \in [0, 1] \) as \( N \to \infty \). Then,

\[ \frac{h_{L(N)}(x_1, \ldots, x_N)}{h_{L(N)}(1, \ldots, 1)} \to \prod_{i=1}^{\infty} \frac{1}{1 - \alpha(x_i - 1)}, \quad (x_1, x_2, \ldots) \in \mathbb{T}_{\text{fin}}^{\infty} \]

Proof. We proceed much as in the proposition above. For a fixed \( K \leq N \), we have

\[ h_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1) \]

\[ = \sum_{\epsilon \in \mathbb{B}_0^K} \#\{ \text{N sequences with sum } L(N) \text{ starting with } (\epsilon_1, \ldots, \epsilon_K) \} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \]

\[ = \sum_{\epsilon \in \mathbb{B}_0^K} \left( \frac{N - K}{L(N) - \sum_{i=1}^{K} \epsilon_i} \right) x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \]

where the last line comes from thinking about

\[ \text{Fill } \sum_{i=1}^{K} \epsilon_i \text{ boxes with } \{1, \ldots, K\} \quad \text{and} \quad \text{Fill } N - \sum_{i=1}^{K} \epsilon_i \text{ boxes with } \{K+1, \ldots, N\} \]
and so

\[
\frac{h_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1)}{h_{L(N)}(1, \ldots, 1)} = \sum_{\epsilon \in N^K \setminus \{\mathbf{0}\}} \left[ \left( \begin{array}{c} N - K \\ L(N) - \sum_{i=1}^K \epsilon_i \end{array} \right) / \left( \begin{array}{c} N \\ L(N) \end{array} \right) \right] x_1^{\epsilon_1} \cdots x_K^{\epsilon_K}
\]

Consider that, for fixed \( K \leq N \), we have

\[
\left( \begin{array}{c} N - K \\ L(N) - \sum_{i=1}^K \epsilon_i \end{array} \right) / \left( \begin{array}{c} N \\ L(N) \end{array} \right) = \frac{(N - K + L(N) - \sum_{i=1}^K \epsilon_i - 1)!}{(N + L(N) - 1)!} \times \frac{(L(N))!}{(L(N) - \sum \epsilon_i)!} \times \frac{(N - 1)!}{(N - K - 1)!}
\]

\[
\approx \frac{(L(N))^{\sum \epsilon_i} N^K}{(N + L(N))^{K + \sum \epsilon_i}}
\]

\[
= \left( \frac{L(N)}{N} \right)^{\sum \epsilon_i} \left( \frac{1}{1 + \frac{L(N)}{N}} \right)^{K + \sum \epsilon_i}
\]

\[
N \to \infty \quad \frac{\alpha}{1 + \alpha} \sum \epsilon_i \left( \frac{1}{1 + \alpha} \right)^K
\]

Thus,

\[
\lim_{N \to \infty} \frac{h_{L(N)}(x_1, \ldots, x_K, 1, \ldots, 1)}{h_{L(N)}(1, \ldots, 1)} = \sum_{\epsilon} \left( \frac{1}{1 + \alpha} \right)^K \left( \frac{\alpha}{1 + \alpha} \right)^{\sum \epsilon_i} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K}
\]

\[
= \prod_{i=1}^K \left( \frac{1}{1 + \alpha} \right) \left( 1 + \frac{\alpha}{1 + \alpha} x_i + \left( \frac{\alpha}{1 + \alpha} \right)^2 x_i^2 + \cdots \right)
\]

\[
= \prod_{i=1}^K \frac{1}{1 + \alpha - \alpha x_i}
\]

So, taking \( K \to \infty \) completes the proof. \( \square \)

REFERENCES