A Raising Operator Formula for Macdonald Polynomials and other related families

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Outline

1. **Background on symmetric functions and Macdonald polynomials**
2. Shuffle theorems, combinatorics, and LLT polynomials
3. A new formula for Macdonald polynomials

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q,t)[x_1, \ldots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

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$$e_r = \sum_{i_1<i_2<\cdots<i_r} x_{i_1}x_{i_2}\cdots x_{i_r} \text{ or } h_r = \sum_{i_1\leq i_2\leq\cdots\leq i_r} x_{i_1}x_{i_2}\cdots x_{i_r}$$

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- E.g. for $n = 3$,

$$\begin{aligned} e_1 &= x_1 + x_2 + x_3 = h_1 \\ e_2 &= x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\ e_3 &= x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \cdots \end{aligned}$$

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- Λ is a $\mathbb{Q}(q,t)$-algebra.

Bases for symmetric functions

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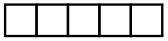
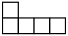
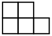
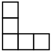
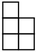
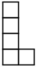
$5 \to$ 

$4 + 1 \to$ 

$3 + 2 \to$ 

$3 + 1 + 1 \to$ 

$2 + 2 + 1 \to$ 

$2 + 1 + 1 + 1 \to$ 

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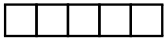
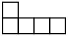
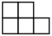
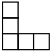
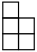
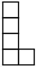
$5 \rightarrow$ 

$4 + 1 \rightarrow$ 

$3 + 2 \rightarrow$ 

$3 + 1 + 1 \rightarrow$ 

$2 + 2 + 1 \rightarrow$ 

$2 + 1 + 1 + 1 \rightarrow$ 

$1 + 1 + 1 + 1 + 1 \rightarrow$ 

$\Longrightarrow$ any basis of symmetric functions is indexed by partitions.

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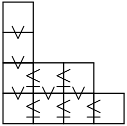

For $\lambda = (2, 1)$,

$$\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}$$

Polynomials from tableaux

Associate a polynomial to SSYT(λ).

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$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} \boldsymbol{x}^T \text{ for } \boldsymbol{x}^T = \prod_{i \in T} x_i$$

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- $\{s_\lambda\}_\lambda$ forms a basis for $\Lambda_{\mathbb{Q}}$.

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- $\text{Ind}_{S_m \times S_n}^{S_{m+n}}(V \times W) \mapsto \text{Frob}(V) \cdot \text{Frob}(W)$
- Upshot: S_n-representations go to symmetric functions in structure preserving way.

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Irreducible representations of S_n are **also** labeled by partitions of n.

Frobenius charactersitc, Frob: $Rep(S_n) \to \Lambda$, such that

- Irreducible S_n-representation V_λ has $\text{Frob}(V_\lambda) = s_\lambda$
- $U \cong V \oplus W \implies \text{Frob}(U) = \text{Frob}(V) + \text{Frob}(W)$
- $\text{Ind}_{S_m \times S_n}^{S_{m+n}}(V \times W) \mapsto \text{Frob}(V) \cdot \text{Frob}(W)$
- Upshot: S_n-representations go to symmetric functions in structure preserving way.

Hidden Guide: Schur Positivity

"Naturally occurring" symmetric functions which are non-negative (coefficients in $\mathbb{N}$) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

An Explicit Example: Harmonic polynomials

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

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$$\begin{aligned} M &= \mathrm{sp}\left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\qquad x_3 - x_1, x_2 - x_3, 1\} \end{aligned}$$

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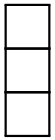


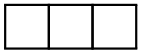

2. How many times does an irreducible S_n-representation occur?
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Harmonic polynomials

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Remark: M is a "regular representation."

Getting more information

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Break M up into smallest S_n fixed subspaces

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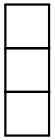

deg=2

deg=1

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Solution: irreducible S_n-representation of polynomials of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\ydiagram{1,1,1}} + q^2 s_{\ydiagram{1,2}} + q^1 s_{\ydiagram{1,2}} + q^0 s_{\ydiagram{3}}$$

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Answer: Hall-Littlewood polynomial $H_{\ydiagram{1,1,1}}(X; q)$.

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- $\tilde{H}_\lambda(X;1,1) = e_1^{|\lambda|}$.
- Does there exist a family of S_n-regular representations whose bigraded Frobenius characteristics equal $\tilde{H}_\lambda(X;q,t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ with $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

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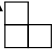
Δ 

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$$\tilde{H}_{(2,1)} = q^1 t^1 s_{(1,1,1)} + t^1 s_{(2,1)} + q^1 s_{(2,1)} + s_{(3)}$$

Garsia-Haiman modules

Theorem (Haiman, 2001)

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- No combinatorial description of $\tilde{K}_{\lambda\mu}(q,t)$.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	$\mathrm{SSYT}(\lambda)$
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	??

Garsia-Haiman modules

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \mathrm{sp}\{f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \mid \left(\sum_{j=1}^{n} \partial_{x_j}^r \partial_{y_j}^s\right) f = 0, \forall r + s > 0\}$$

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Question

What symmetric function is the bigraded Frobenius characteristic of DH_n?

∇e_n

Frobenius characteristic of DH_3

∇e_n

Frobenius characteristic of DH_3

$$= \frac{t^3\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2t+qt^2+qt)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{q^3\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

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Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q+t+1)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

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Operator ∇

$$\nabla \tilde{H}_\lambda(X;q,t) = q^{n(\lambda)} t^{n(\lambda^*)} \tilde{H}_\lambda(X;q,t)\,,$$

where $n(\lambda) = \sum_i (i-1)\lambda_i$ and λ^* is the transpose partition to λ.

∇e_n

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Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n.

Symmetric functions, representation theory, and combinatorics

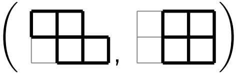
Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	SSYT(λ)
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	??
∇e_n	DH_n	Shuffle theorem

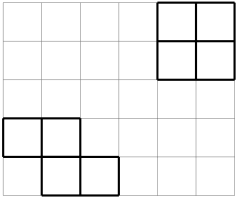
Outline

1. Background on symmetric functions and Macdonald polynomials
2. **Shuffle theorems, combinatorics, and LLT polynomials**
3. A new formula for Macdonald polynomials

Key Object: LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)

$\boldsymbol{\nu} =$ 



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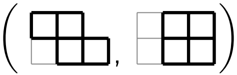
$\boldsymbol{\nu} =$ 

-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

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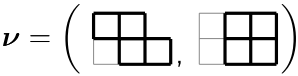

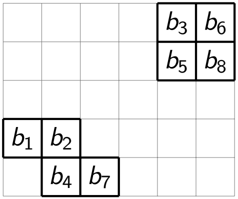

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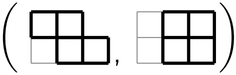
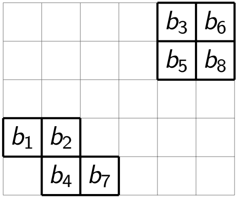
$\boldsymbol{\nu} =$ 



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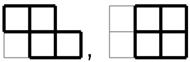
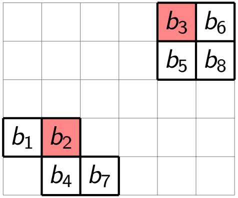
$\boldsymbol{\nu} = ($  $)$



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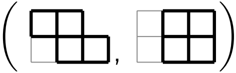
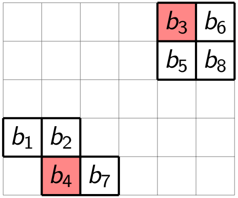
$\boldsymbol{\nu} =$ 



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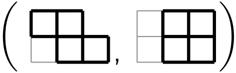
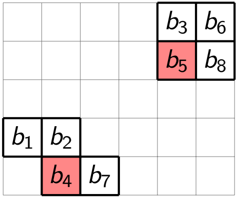
$\boldsymbol{\nu} =$ 



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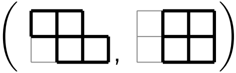
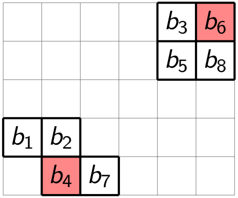
$\boldsymbol{\nu} =$ 



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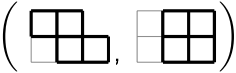
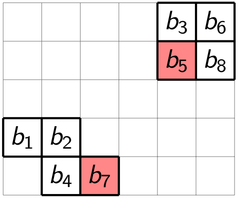
$\boldsymbol{\nu} =$ 



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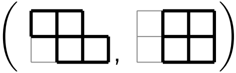
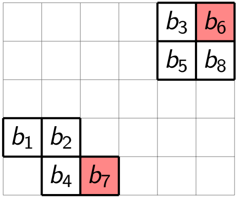
$\boldsymbol{\nu} =$ 



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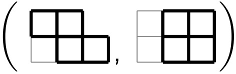
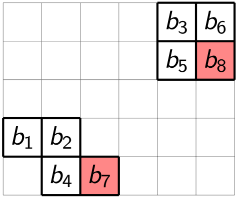
$\boldsymbol{\nu} =$ 



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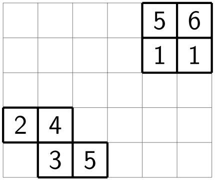

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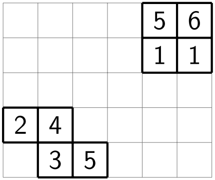

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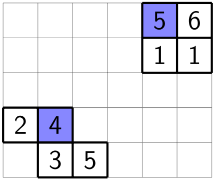

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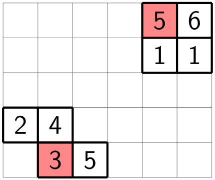

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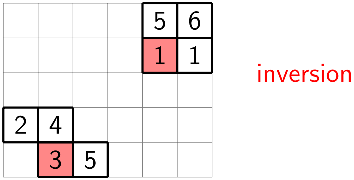

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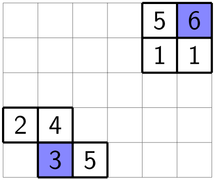

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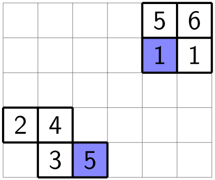

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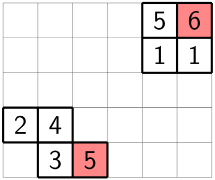

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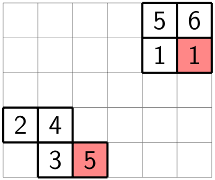

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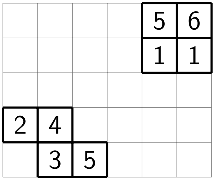

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- $\mathcal{G}_{\boldsymbol{\nu}}$ is Schur-positive for any tuple of skew shapes $\boldsymbol{\nu}$ [Grojnowski-Haiman, 2007].

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} (q,t \;\; monomial)(LLT \;\; polynomial)$$

- Summation over all k-by-k Dyck paths.

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} (\mathit{LLT\ polynomial})$$

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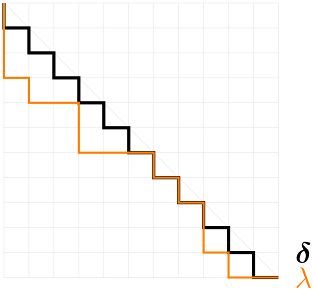
- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



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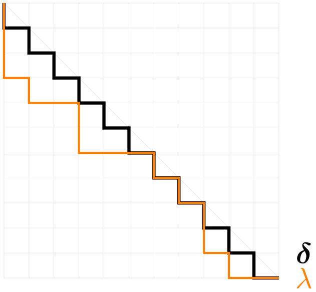

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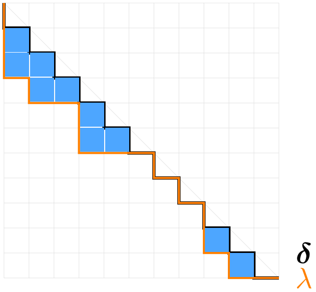

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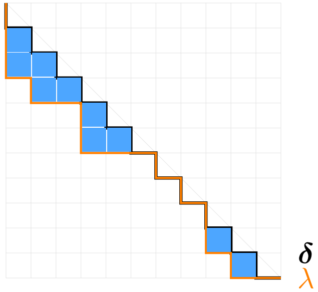

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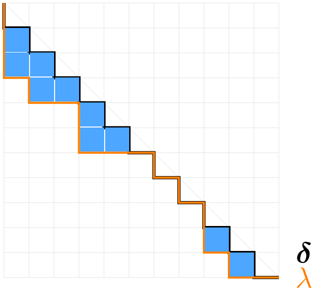

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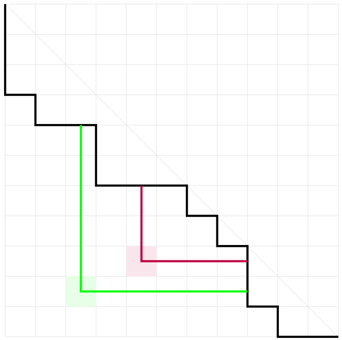
dinv

dinv(λ) $=\#$ of balanced hooks in diagram below λ.



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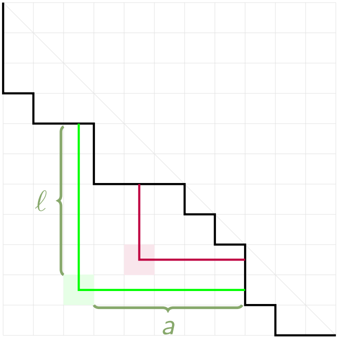

Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1-\epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

Example ∇e_3

λ	$q^{\mathsf{dinv}(\lambda)} t^{\mathsf{area}(\lambda)}$	$q^{\mathsf{dinv}(\lambda)} t^{\mathsf{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$

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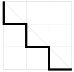		
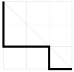		
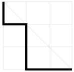		
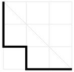		
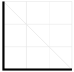		

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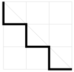
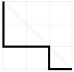
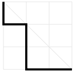
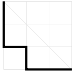
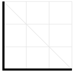
	q^3	
	$q^2 t$	
	qt	
	qt^2	
	t^3	

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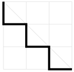
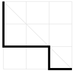
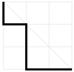
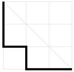
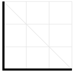
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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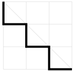
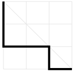
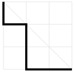
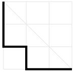
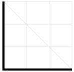
	q^3	$s_3 + q s_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$q t s_{2,1} + q^2 t s_{1,1,1}$
	$q t$	$t s_{2,1} + q t s_{1,1,1}$
	$q t^2$	$t^2 s_{2,1} + q t^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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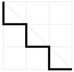
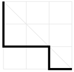
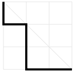
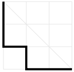
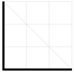
	q^3	$s_3 + q s_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$q t s_{2,1} + q^2 t s_{1,1,1}$
	qt	$t s_{2,1} + q t s_{1,1,1}$
	qt^2	$t^2 s_{2,1} + q t^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

- Entire quantity is q, t-symmetric
- Coefficient of $s_{1,1,1}$ in sum is a "(q,t)-Catalan number" $(q^3 + q^2 t + qt + qt^2 + t^3)$.

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When a problem is too difficult, try generalizing!

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Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2021)

For $m, n > 0$ coprime, the operator $e_k^{(m,n)}$ acting on Λ satisfies

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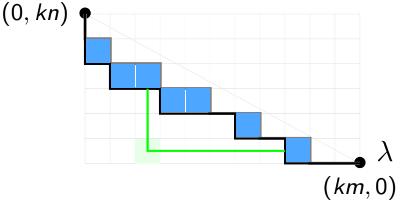

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Algebra $\mathcal{E} \curvearrowright \Lambda =$ symmetric polynomials

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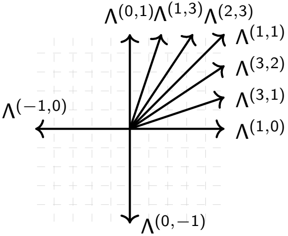

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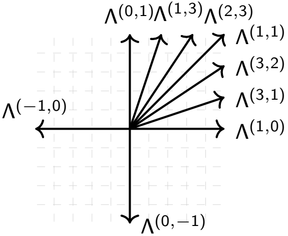

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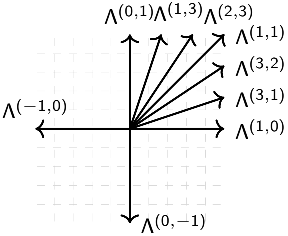

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LHS of Shuffle Theorem $= e_k^{(1,1)} \in \Lambda^{(1,1)}$ acting on $1 \in \Lambda$.
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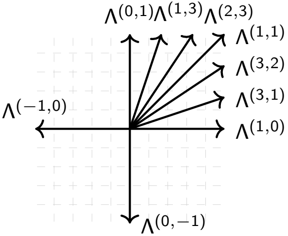

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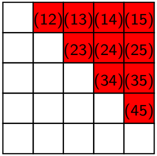
Can be difficult to work with in general. Can we make it more explicit?

Root ideals

$R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$ denotes the set of positive roots for GL_n, where $\alpha_{ij} = \epsilon_i - \epsilon_j$.



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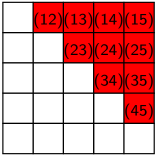
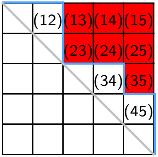


A root ideal $\Psi \subseteq R_+$ is an upper order ideal of positive roots.



$\Psi =$ Roots above Dyck path

Schur functions revisited

- Convention: $h_0 = 1$ and $h_d = 0$ for $d < 0$.
- For any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$, set

$$s_\gamma = \det(h_{\gamma_i + j - i})_{1 \leq i,j \leq n}$$

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Then, $s_\gamma = \pm s_\lambda$ or 0 for some partition λ.

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- For any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$, set

$$s_\gamma = \det(h_{\gamma_i+j-i})_{1\le i,j\le n}$$

Then, $s_\gamma = \pm s_\lambda$ or 0 for some partition λ.
Precisely, for $\rho = (n-1, n-2, \ldots, 1, 0)$,

$$s_\gamma = \begin{cases} \operatorname{sgn}(\gamma+\rho) s_{\text{sort}(\gamma+\rho)-\rho} & \text{if } \gamma+\rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

- $\text{sort}(\beta)$ = weakly decreasing sequence obtained by sorting β,
- $\operatorname{sgn}(\beta)$ = sign of the shortest permutation taking β to $\text{sort}(\beta)$.

Example: $s_{201} = 0, s_{2\text{-}11} = -s_{200}$.

Weyl symmetrization

Define the *Weyl symmetrization operator* $\boldsymbol{\sigma}\colon \mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \to \Lambda(X)$ by linearly extending

$$\boldsymbol{z}^{\gamma} \mapsto s_{\gamma}(X)$$

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Example

$\boldsymbol{\sigma}(\boldsymbol{z}^{111} + \boldsymbol{z}^{201} + \boldsymbol{z}^{210} + \boldsymbol{z}^{3\text{-}11}) = s_{111} + s_{201} + s_{210} + s_{3\text{-}11} = s_{111} + s_{210} - s_{300}$

Catalanimals

Definition

The *Catalanimal* indexed by $R_q, R_t, R_{qt} \subseteq R_+$ and $\lambda \in \mathbb{Z}^n$ is

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where $\mathbf{z}^{\alpha_{ij}} = z_i/z_j$ and $(1 - tz_i/z_j)^{-1} = 1 + tz_i/z_j + t^2 z_i^2/z_j^2 + \cdots$.

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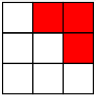
With $n = 3$, $R_+ =$ 

$$H(R_+, R_+, \{\alpha_{13}\}, (111)) =$$

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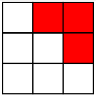
With $n = 3$, $R_+ =$ 

$$H(R_+, R_+, \{\alpha_{13}\}, (111)) = \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^{111}(1 - qtz_1/z_3)}{\prod_{1 \le i < j \le 3}(1 - qz_i/z_j)(1 - tz_i/z_j)}\right)$$

$$= s_{111} + (q + t + q^2 + qt + t^2)s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3)s_3$$

$$= \omega \nabla e_3.$$

Why?

Let $R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq l\}$ and $R_+^0 = \{\alpha_{ij} \in R_+ \mid i + 1 < j\}$.

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$$e_k^{(m,n)} \cdot 1 = H(R_+, R_+, R_+^0, \mathbf{b})$$

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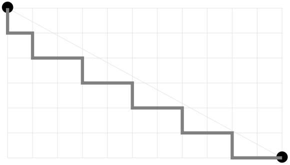
for $\mathbf{b} = (b_0, \ldots, b_{km-1})$ satisfying b_i = the number of south steps on vertical line $x = i$ of highest lattice path under line $y + \frac{n}{m}x = n$.

δ = highest Dyck path.



δ

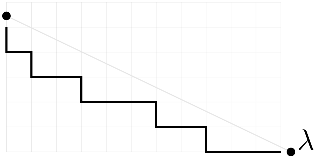
$$\mathbf{b} = (1,1,0,1,0,1,0,1,0,1,0)$$

Results

Manipulating Catalanimal $\Longrightarrow$ a proof of the Rational Shuffle Theorem + a generalization.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $\mathbf{b} = (b_1, \ldots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.



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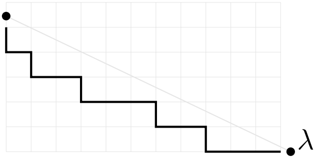

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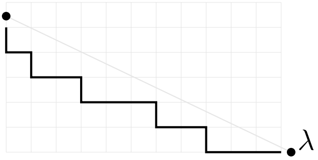

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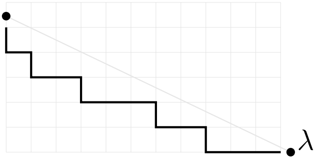
$$H(R_+, R_+, R_+^0, \mathbf{b}) = \sum_{\lambda} \qquad \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line $y + px = s$,



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$$H(R_+, R_+, R_+^0, \mathbf{b}) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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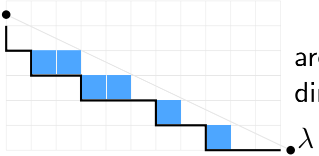

area(λ) as before

$\text{dinv}_p(\lambda) = \#p$-balanced hooks $\frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

A Question

Why stop at $e_k^{(m,n)}$?

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Special case: $\mathcal{G}_{\boldsymbol{\nu}}^{(1,1)} \cdot 1 = \nabla \mathcal{G}_{\boldsymbol{\nu}}(X; q)$.

LLT Catalanimals

For a tuple of skew shapes $\boldsymbol{\nu}$, the *LLT Catalanimal* $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ is determined by

- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$,

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- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$,
- $R_+ \setminus R_q =$ pairs of boxes in the same diagonal in the same shape,
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- λ: fill each diagonal D of $\boldsymbol{\nu}$ with
  $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.
  Listing this filling in reading order gives λ.

LLT Catalanimals

- ■ $R_+ \setminus R_q$ = pairs of boxes in the same diagonal,
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- ■ R_{qt} = all other pairs,

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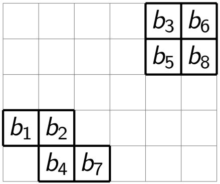
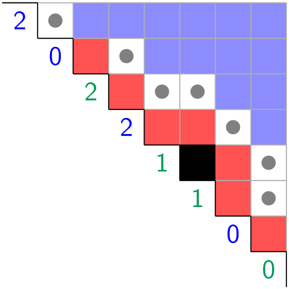


$\boldsymbol{\nu}$



LLT Catalanimals

- (black square) $R_+ \setminus R_q$ = pairs of boxes in the same diagonal,
- (red square) $R_q \setminus R_t$ = the attacking pairs,
- (grey dot) $R_t \setminus R_{qt}$ = pairs going between adjacent diagonals,
- (blue square) R_{qt} = all other pairs,

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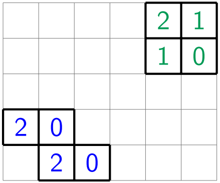
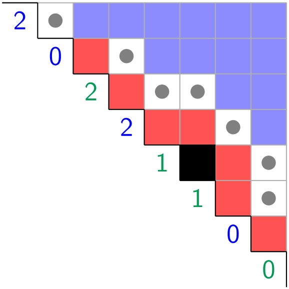
$1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.



λ, as a filling of $\boldsymbol{\nu}$



LLT Catalanimals

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021+)

Let $\boldsymbol{\nu}$ be a tuple of skew shapes and let $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ be the associated LLT Catalanimal. Then

$$\begin{aligned}\nabla \mathcal{G}_{\boldsymbol{\nu}}(X;q) &= c_{\boldsymbol{\nu}}\, \omega H_{\boldsymbol{\nu}} \\ &= c_{\boldsymbol{\nu}}\, \omega \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^{\lambda} \prod_{\alpha \in R_{qt}} (1 - qt\, \boldsymbol{z}^{\alpha})}{\prod_{\alpha \in R_q} (1 - q\, \boldsymbol{z}^{\alpha}) \prod_{\alpha \in R_t} (1 - t\, \boldsymbol{z}^{\alpha})} \right)\end{aligned}$$

for some $c_{\boldsymbol{\nu}} \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$.

What about Macdonald polynomials?!

- Remember $\nabla \tilde{H}_\mu = q^{n(\mu)} t^{n(\mu^*)} \tilde{H}_\mu$.

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Outline

1. Background on symmetric functions and Macdonald polynomials
2. Shuffle theorems, combinatorics, and LLT polynomials
3. **A new formula for Macdonald polynomials**

Haglund-Haiman-Loehr formula example

$$\tilde{H}_\mu(X;q,t) = \sum_D \left(\prod_{u\in D} q^{-\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1}\right) \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}(X;q)$$

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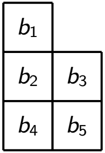
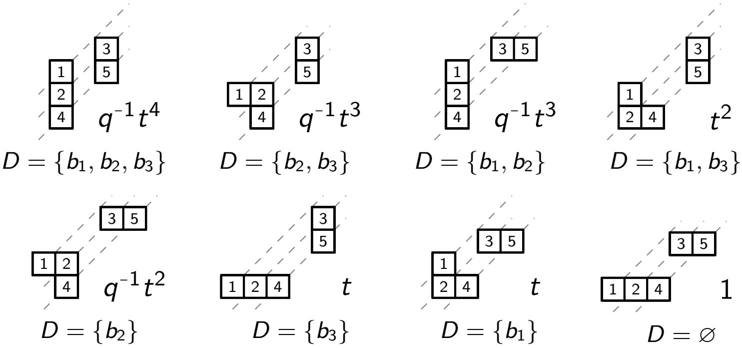


μ



Putting it all together

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D}\mathcal{G}_{\boldsymbol{\nu}(\mu,D)}$ and apply $\omega\nabla$.

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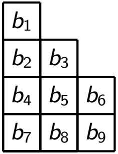
- By construction, all the LLT Catalanimals $H_{\boldsymbol{\nu}(\mu,D)}$ appearing on the RHS will have the same root ideal data (R_q, R_t, R_{qt}).
- Collect terms to get $\prod_{(b_i,b_j)\in V(\mu)}(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j)$ factor for $V(\mu)$ the set of vertical dominoes (b_i, b_j) in μ.

$$\tilde{H}_\mu = \omega\boldsymbol{\sigma}\left(z_1 \cdots z_n \frac{\prod_{\alpha_{ij}\in V(\mu)} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod_{\alpha\in \widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+}\left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha\in R_\mu}\left(1 - t\boldsymbol{z}^\alpha\right)} \right).$$

The root ideal R_μ



row reading order

$b_1 \prec b_2 \prec \cdots \prec b_n$

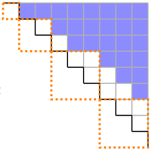
$$R_\mu := \{\alpha_{ij} \in R_+ \mid \mathrm{south}(b_i) \preceq b_j\},$$

$$\widehat{R}_\mu := \{\alpha_{ij} \in R_+ \mid \mathrm{south}(b_i) \prec b_j\},$$

$$R_\mu \setminus \widehat{R}_\mu \leftrightarrow V(\mu) = \text{vertical dominoes in } \mu$$

Example:

$R_{3321} =$



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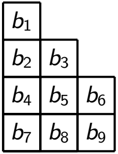

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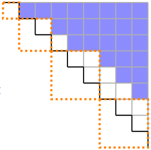
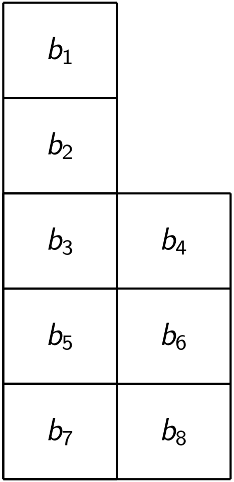


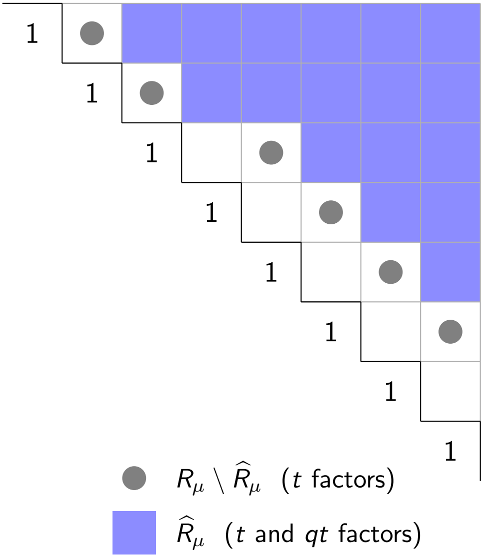
Remark

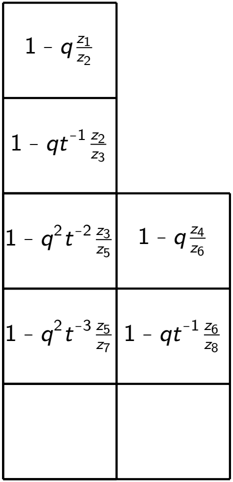
$$\tilde{H}_\mu(X;0,t) = \omega\boldsymbol{\sigma}\left(\frac{z_1\cdots z_n}{\prod_{\alpha\in R_\mu}(1-t\boldsymbol{z}^\alpha)}\right)$$

Example

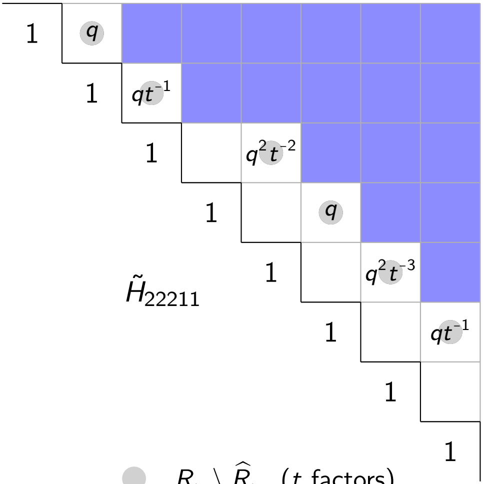


partition $\mu = 22211$





numerator factors $1 - q^{\mathrm{arm}+1}t^{-\mathrm{leg}}z_i/z_j$



$R_\mu \setminus \widehat{R}_\mu$ (t factors)

$\widehat{R}_\mu$ (t and qt factors)

$q = t = 1$ specialization

$$
\begin{aligned}
&\omega\boldsymbol{\sigma}\left(z_1\cdots z_n \frac{\displaystyle\prod_{\alpha_{ij}\in R_\mu\setminus\widehat{R}_\mu}\left(1-q^{\mathrm{arm}(b_i)+1}t^{-\mathrm{leg}(b_i)}z_i/z_j\right)\prod_{\alpha\in\widehat{R}_\mu}\left(1-qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+}\left(1-q\boldsymbol{z}^\alpha\right)\prod_{\alpha\in R_\mu}\left(1-t\boldsymbol{z}^\alpha\right)}\right)\\
\overset{q=t=1}{\to}&\omega\boldsymbol{\sigma}\left(z_1\cdots z_n \frac{\prod_{\alpha\in R_\mu\setminus\widehat{R}_\mu}(1-\boldsymbol{z}^\alpha)\prod_{\alpha\in\widehat{R}_\mu}(1-\boldsymbol{z}^\alpha)}{\prod_{\alpha\in R_+}(1-\boldsymbol{z}^\alpha)\prod_{\alpha\in R_\mu}(1-\boldsymbol{z}^\alpha)}\right)\\
=&\omega\boldsymbol{\sigma}\left(\frac{z_1\cdots z_n}{\prod_{\alpha\in R_+}(1-\boldsymbol{z}^\alpha)}\right)\\
=&\omega h_1^n\\
=&e_1^n
\end{aligned}
$$

A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

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What can this formula tell us that other formulas for Macdonald polynomials do not?

$$\tilde{H}^{(s)}_\mu := \omega \boldsymbol{\sigma} \left((z_1 \cdots z_n)^s \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod_{\alpha \in \widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_+} \left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha \in R_\mu} \left(1 - t\boldsymbol{z}^\alpha\right)} \right)$$

Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition μ and positive integer s, the symmetric function $\tilde{H}^{(s)}_\mu$ is Schur positive. That is, the coefficients in

$$\tilde{H}^{(s)}_\mu = \sum_\nu K^{(s)}_{\nu,\mu}(q,t)\, s_\nu(X)$$

satisfy $K^{(s)}_{\nu,\mu}(q,t) \in \mathbb{N}[q,t]$.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	$\mathrm{SSYT}(\lambda)$
$\tilde{H}_\lambda(X;q,t)$	Garsia-Haiman M_λ	HHL
∇e_n	DH_n	Shuffle theorem
$\tilde{H}_\lambda^{(s)}(X;q,t)$	??	??

Thank you!

Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. 2023/ed. *A Shuffle Theorem for Paths Under Any Line*, Forum of Mathematics, Pi **11**, e5, DOI 10.1017/fmp.2023.4.

______. 2024. *LLT Polynomials in the Schiffmann Algebra*, Journal für die reine und angewandte Mathematik (Crelles Journal) **811**, 93–133, DOI 10.1515/crelle-2024-0012.

______. 2025. *A Raising Operator Formula for Macdonald Polynomials*, Forum of Math, Sigma **13**, e47, DOI 10.1017/fms.2025.8.

Burban, Igor and Olivier Schiffmann. 2012. *On the Hall algebra of an elliptic curve, I*, Duke Math. J. **161**, no. 7, 1171–1231, DOI 10.1215/00127094-1593263. MR2922373

Carlsson, Erik and Mellit, Anton. 2018. *A Proof of the Shuffle Conjecture* **31**, no. 3, 661–697, DOI 10.1090/jams/893.

Feigin, B. L. and Tsymbaliuk, A. I. 2011. *Equivariant K-theory of Hilbert Schemes via Shuffle Algebra*, Kyoto J. Math. **51**, no. 4, 831–854.

Garsia, Adriano M. and Mark Haiman. 1993. *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A. **90**, no. 8, 3607–3610, DOI 10.1073/pnas.90.8.3607. MR1214091

Haglund, J., M. Haiman, and N. Loehr. 2005. *A Combinatorial Formula for Macdonald Polynomials* **18**, no. 3, 735–761 (electronic).

Haglund, J. and Haiman, M. and Loehr. 2005. *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J. **126**, no. 2, 195–232, DOI 10.1215/S0012-7094-04-12621-1.

Haiman, Mark. 2001. *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14**, no. 4, 941–1006, DOI 10.1090/S0894-0347-01-00373-3. MR1839919

______. 2002. *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149**, no. 2, 371–407, DOI 10.1007/s002220200219. MR1918676

Lascoux, Alain, Bernard Leclerc, and Jean-Yves Thibon. 1995. *Ribbon tableaux, Hall-Littlewood functions and unipotent varieties*, Sém. Lothar. Combin. **34**, Art. B34g, approx. 23. MR1399754

Mellit, Anton. 2021. *Toric Braids and (m,n)-Parking Functions*, Duke Math. J. **170**, no. 18, 4123–4169, DOI 10.1215/00127094-2021-0011.

Negut, Andrei. 2014. *The shuffle algebra revisited*, Int. Math. Res. Not. IMRN **22**, 6242–6275, DOI 10.1093/imrn/rnt156. MR3283004

Schiffmann, Olivier and Vasserot, Eric. 2013. *The Elliptic Hall Algebra and the K-theory of the Hilbert Scheme of A2*, Duke Mathematical Journal **162**, no. 2, 279–366, DOI 10.1215/00127094-1961849.